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Includes Formulas for Gems

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SPUR GEARS

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DESIGN, OPERATION, AND PRODUCTION

BY

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PREFACE

The subject of gearing is an old one, yet our knowledge of it is still far from complete. New and more severe requirements that gearing must meet are developing continually. New and improved materials and methods for the production of gears are also being made available. To a great extent, however, the design of gear-tooth profiles has become conventionalized, and many of the fundamentals of gear-tooth action have been lost sight of in this process. As a result, the full benefits of the improved materials and methods of production have not always been secured.

The introduction of the formed cutter for the production of gear teeth imposed upon the manufacturers of these small tools the task of gear-tooth design. As a matter of fact, this was a problem that the majority of other manufacturers was glad to shift to someone else. Certain conventions based upon the limitations of such formed tools were thus introduced into gear-tooth design. Many of such conventions were naturally carried over into the design of gear-tooth forms produced by molding or generating processes, although the limitations of the molding processes are quite different, in many respects, from those of form cutting. For example, as a matter of economy, which is always a controlling one in all industrial processes, the number of different formed cutters required to produce mating gears of all different tooth numbers should be reduced to a minimum. This has required the use of fixed tooth proportions for all gears of any one series or system. With the molding processes, however, and the use of the full-involute form for the gear-tooth profiles, a single cutter will produce mating gears without the limitation of fixed tooth proportions. In general, economy of manufacture of gears requires the standardization and simplification of the tools used to produce the gear teeth rather than the standardization of the gears themselves.

The purpose of this book is to bring out as clearly and simply as possible the fundamental characteristics of spur gears, in the hope that more effective use may be made of the facilities now

available for producing them. The attempt has been made to give a complete mathematical exposition of this subject as simply as possible and at the same time to include in the text sufficient explanation, so that a grasp of the subject may also be gained without following through all of the mathematical proofs. To this end, many tables have been included to simplify the use of the material.

The author can claim but little originality on his part for the material published here. His major task has been the selection and the arrangement of the work of many others into such a reference book on this subject as he would desire for himself. In order to make it more nearly complete, certain assumptions are used where exact knowledge is lacking. Where assumptions are made, however, they are plainly stated as such. The author has tried to give due credit in all cases where the original source of the information he has used was known; there are undoubtedly many other cases where this, because of lack of definite information, has not been done.

This book is primarily one on the subject of involute spur-gear teeth. To the author's mind, this subject is naturally divided into three main branches: first, their design; second, their operation; and third, their production. The book is therefore divided into three such sections.

EARLE BUCKINGHAM

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SECTION 1

THE DESIGN OF GEAR-TOOTH FORMS

SPUR GEARS

CHAPTER I

CONJUGATE GEAR TOOTH ACTION

There is an almost infinite number of forms which can be used as gear-tooth profiles. The essential purpose of gear-tooth profiles is to transmit rotary motion from one shaft to another. Usually the additional requirement of uniform motion also exists.

Although the involute profile is the one most commonly used today for gear-tooth forms, occasions arise when some other form of profile must be employed. In addition, there are also other problems than the transmission of rotary motion where a thorough knowledge of the theory of gearing assists in the most direct solution. One of such problems is the hobbing of spline shafts.

Again, in order to appreciate fully the great simplicity of the involute form, both in theory and in its production, it is necessary to have a clear understanding of the principles of conjugate gear-tooth action. We will therefore consider at this time the characteristics of tooth profiles that will transmit through each other uniform rotary motion. The action between such profiles is called *conjugate gear-tooth action.*

In essence, a pair of mating gear-tooth profiles are cams, the one acting against the other to produce the desired relative motion. With certain restrictions, one profile can be chosen at random, and a proper mating profile can be developed.

As a definite example, we will now consider the action between two lever arms, the one with a pin which engages in a slot of the other, the driven member. This is illustrated in Fig. 1.

The lever with the pin is rotating with uniform motion in the direction shown by the arrow. The rate of rotary motion of the driven or slotted lever depends upon the relative lengths of a and b, as shown in the illustration. The driven lever will move b/a times as fast as the driver. If these lengths a and b are equal

and remain equal in all contact positions, the rates of rotation will also be equal.

The lengths of a and b are determined by the intersection of the normal to the mating profiles at the point of contact with the common center line of the axes of the two levers. A *normal* is a

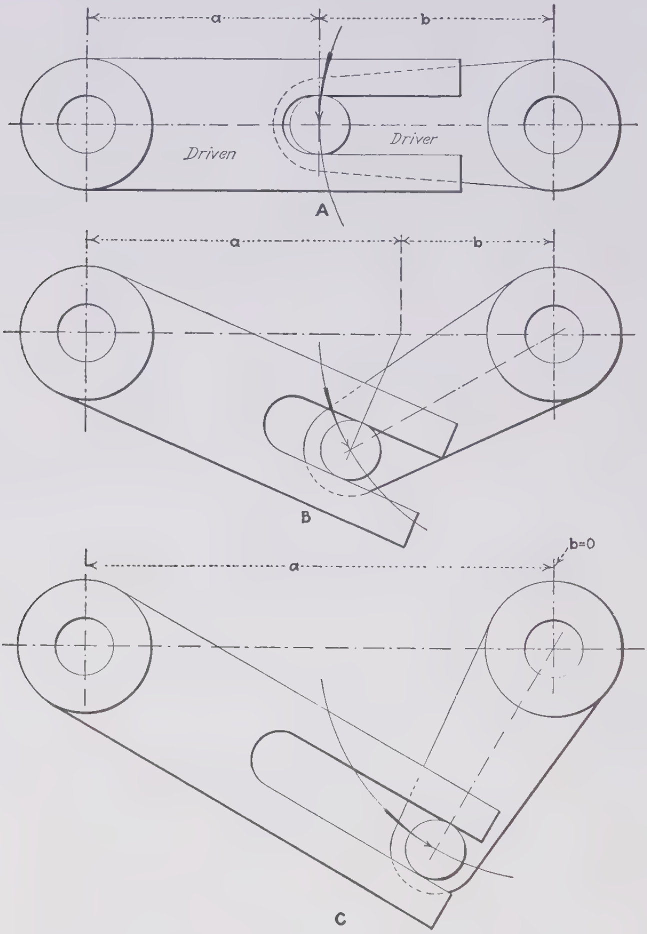

FIG. 1.—Transmission of rotary action through lever arms.

line which is perpendicular to the tangent of a curve at its point of tangency. Thus, the normal to a straight line is the perpendicular to it, and the normal to a circle is a radial line.

At the top of Fig. 1, position A, the distances a and b are shown as equal. At this position, therefore, the rates of rotation of the

two levers are equal. At position B, when the driver has moved 30 deg. from its original position, b is shorter and a is longer than originally; therefore, the driven or slotted lever is moving more slowly than the driver. At position C, when the driver has moved 60 deg. from its original position, the length of b has become zero, while the length of a has become double its original length. At this position, the driven lever has ceased to move. Further motion of the driver, assuming a sufficient length of slot in the driven lever, would cause the driven lever to start to move in the opposite direction.

It is evident that the action between the two levers, shown in Fig. 1, does not result in the transmission of uniform rotary motion from one shaft to the other. In order to transmit this uniform motion, the relative lengths of a and b must remain constant at all operating positions of the mating profiles. This gives us the basic law of conjugate gear-tooth profiles, which may be expressed as follows:

To transmit uniform rotary motion from one shaft to another by means of gear teeth, the normals to the profiles of these teeth at all points of contact must pass through a fixed point in the common center line of the two shafts.

This fixed point in the common center line is called the *pitch point*. With every gear-tooth form, except the involute, there is a definite pitch line or circle from which the conjugate tooth profiles must be developed. The pitch circles of mating gears must be tangent to each other. The point of tangency of these pitch circles is the pitch point. These pitch circles are of such size that if they were to drive each other by friction, they would transmit the required relative motion. The sizes of these pitch circles are inversely proportional to the rate of rotation; for equal speed, these sizes are equal; for double speed, the pitch circle of the slower gear is twice the size of the faster, etc. The tooth profile may be symmetrical or unsymmetrical in respect to the pitch line; it may be all above it or all below it or partly above and partly below.

As stated before, the profile of one gear may be chosen arbitrarily, and the conjugate profile for the mating gear can be developed. For every conjugate gear-tooth profile there is also a basic-rack form. As mating tooth profiles act together, the point of contact between them will travel along a line or path called the *line of action* or the *path of contact*. Once a pitch line

has been established for any profile, a definite line of action exists, along which contact with all other conjugate profiles is made. There is a definite relation between a gear-tooth profile and its line of action, so that if either one is given, the other is fixed. When the tooth profile is given, it is a simple matter to construct its line of action in regard to any given pitch line, but when the line of action alone is given, it is a much more difficult task to construct the tooth profile. If, in addition to the line of action, we definitely know certain points on it that represent known angular or linear movements of the profile, this problem is greatly simplified.

In order to clarify the foregoing, we will consider several definite problems. With the exception of a few mathematical curves, such problems must be solved geometrically or by actual layouts. In practice, these are made to a sufficiently enlarged scale to obtain the desired degree of accuracy. The following examples indicate the methods employed in making such layouts:

PROBLEM 1.—*Given an arbitrary gear-tooth profile, to construct its line of action.*

The arbitrary tooth profile selected is a straight line, as shown in Fig. 2. The pitch line is established at the middle of the tooth height. This gear has 36 teeth, giving an angular distance of 10 deg. between the successive teeth.

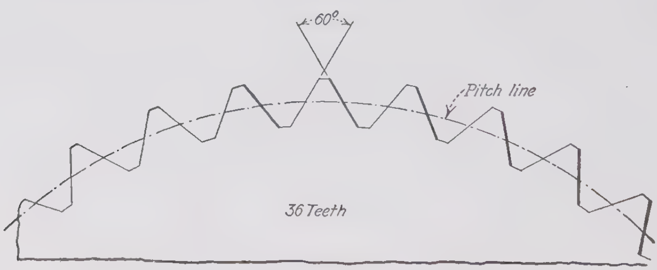

FIG. 2.—Straight-line gear-tooth profile.

To construct the line of action, the tooth profile is rotated about its axis into several successive positions. At each position, a line through the pitch point and perpendicular to the tooth profile is drawn. The intersection of this line with its respective tooth profile is a point on the line of action. This point is where contact is made with any mating conjugate profile with the given profile in this position. After determining a series of such points,

the line of action can be drawn through them. It is usually a good plan when laying out the successive positions of the tooth profile to make the intervals equal, as this greatly assists later in the construction of mating conjugate profiles.

The construction of the line of action for the selected profile is shown in Fig. 3. On the left, at *A*, are shown the construction lines only, that is, the successive positions of the tooth profile and the perpendiculars through the pitch point to each position of the tooth profile. The intersections, or contact points, are marked with heavy dots. Each position of the tooth profile is numbered for identification later. In this example, the intervals between the successive positions of the tooth profile are equal to 5 deg., or one-half the tooth spacing. In actual practice, the scale of the drawing would be greatly enlarged, and the intervals between the successive positions of the tooth profile would be much smaller.

At the right, in Fig. 3, at *B*, the line of action is shown in a heavy line. A study of this illustration should make clear the method of constructing the line of action from a given gear-tooth profile. When the profile is not a straight line, the tangents and normals to the curved profile must first be determined; otherwise, the process is identical. Usually, the arbitrary profiles are straight lines, arcs of circles, or a combination of the two.

PROBLEM 2.—*Given an arbitrary gear-tooth profile, to construct the profile of its basic rack.*

The arbitrary tooth profile will be the same as before, illustrated in Fig. 2. To construct the profile of the basic rack, a series of contact points must first be determined, as was done to construct the line of action. The pitch line on the rack will be a straight line. A straight line representing this pitch line is drawn, and on it are laid off intervals corresponding in length to the length of the arcs on the pitch circle of the gear between the successive angular positions of the gear-tooth profile. In this example, the lengths are equal to an arc of 5 deg. on the pitch circle of the original gear. The construction is shown in Fig. 4. At *A* is shown the same layout as at *A* in Fig. 3. The intervals along the pitch line of the rack at *B* are numbered in the reverse order to the successive positions of the profile at *A* and represent the successive positions of the pitch point as the original gear is rotated to the corresponding positions. By drawing a series of parallelograms, successive points on the basic-rack profile are

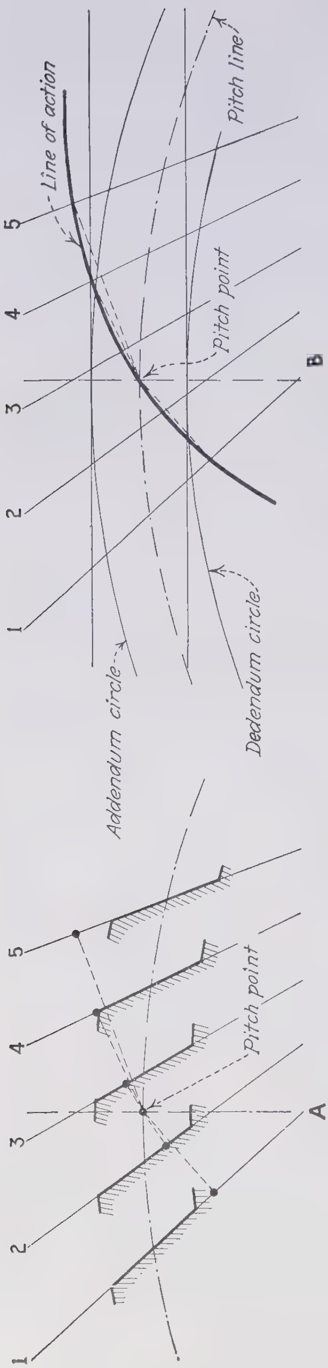

FIG. 3.—Development of the line of action of a straight-line gear tooth.

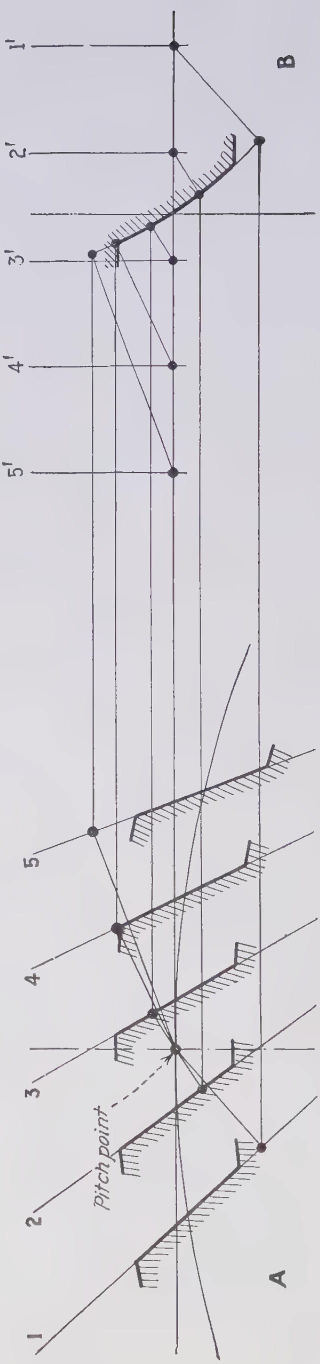

FIG. 4.—Development of the basic rack of a straight-line gear tooth.

determined. From pitch point 1′ for the rack a line is drawn parallel to the normal line to the gear profile at position 1. From contact point 1 on the gear profile, a line parallel to the pitch line of the rack is drawn. The intersection of these two lines gives the first point of the basic-rack profile. Other successive points of the basic-rack profile are determined in a similar manner.

The cross-hatching indicates which side of this basic-rack form would be solid when meshing with the original gear. The basic rack to mesh with the conjugate gear would be reversed; that is, the cross hatching would be on the other side of the line representing the basic form. The form of this line, however, is unchanged. A study of Fig. 4 should make clear the process of constructing the basic-rack form from a given profile.

PROBLEM 3.—*Given an arbitrary gear-tooth profile, to construct the profile of its conjugate gear tooth.*

The arbitrary tooth profile will be the same as that shown in Fig. 2. The construction of the mating conjugate tooth form is very similar to the construction of the basic-rack profile, with the difference that angular intervals and radial distances are used instead of linear intervals and straight lines. A pitch diameter of the desired size is first drawn, and angular intervals corresponding to those on the original gear are laid off. If the pitch diameter of the conjugate gear is to be the same as that of the original gear, the angular intervals will also be the same on both gears. If the pitch diameter of the conjugate gear is to be twice that of the original, the angular intervals will be one-half of those on the original gear, etc. In other words, the lengths of the arcs of these intervals on the pitch lines must always be the same.

In the example shown in Fig. 5, the pitch diameter is made the same as that of the original gear. The intervals, therefore, are 5 deg., the same as before. At A is shown the same layout as at A in Fig. 3 but turned upside down for convenience in constructing the conjugate profile. The intervals on the pitch line of the conjugate gear are numbered in the reverse order to the successive positions of the profile at A and represent the successive positions of the pitch point as the original gear is rotated to the corresponding positions. Instead of drawing a series of parallelograms as for the basic-rack form, we construct a series of similar triangles to determine the conjugate gear-tooth form. From pitch point 1′ on the conjugate gear an arc is drawn equal in

radius to the distance from the pitch point at A to contact point 1. From the center of the conjugate gear another arc is drawn equal in radius to the distance from the corresponding center at A to the contact point 1. The intersection of these two arcs gives the first point on the profile of the mating gear-tooth form. Other successive points of the conjugate gear-tooth profile are determined in a similar manner. A study of Fig. 5 should make clear the process of constructing the conjugate gear-tooth profile from any given gear-tooth profile.

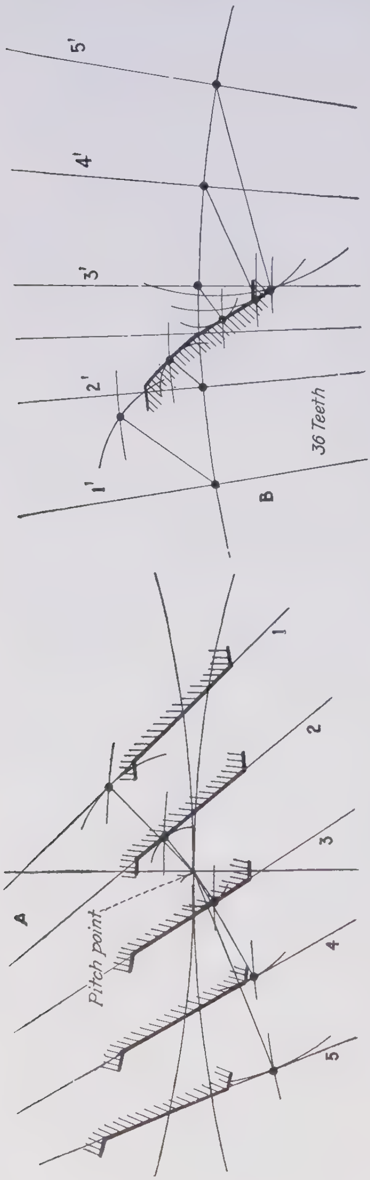

Fig. 5.—Construction of the profile of a conjugate gear tooth of a straight-line gear tooth.

Separate drawings have thus far been made to construct the line of action, the profiles of the basic rack, and the conjugate gear-tooth profile. In actual practice, separate drawings are not necessarily required. These were made here so that the several operations would be more apparent.

Problem 4.—*Given an arbitrary rack profile, to construct the line of action and conjugate gear-teeth forms.*

The arbitrary rack form selected will be the arc of a circle, as shown in Fig. 6. To construct the line of action, lay off even intervals along the pitch line of the rack. Through these points, draw the profile of the rack. As this profile is the arc of a circle,

the centers of these profiles must also be established. These successive profiles will be numbered as before to identify them, and the corresponding centers of the profiles will also be numbered accordingly. From the pitch point, lines are drawn normal to the rack profile in each of its successive positions. As these profiles are arcs of a circle, the normal line will be one which passes through the center of the circle. The intersection of each normal with its rack profile establishes a point of contact, or one point on the line of action. The line drawn through all of these intersection points is the line of action. This has been done at *A* in Fig. 7. A study of this illustration should make clear the process of constructing the line of action for any rack form.

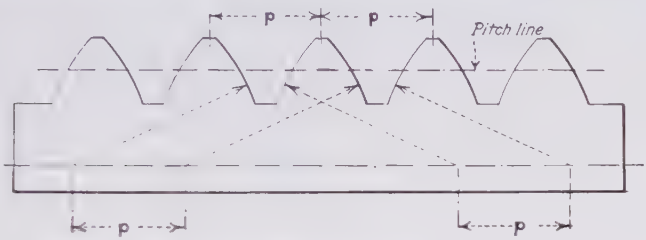

FIG. 6.—Rack-tooth profile constructed with circular arcs.

The construction of the mating conjugate gear-tooth profiles is shown at *B* in Fig. 7. Pitch diameters of the desired size are drawn, and angular intervals, whose lengths of arcs on these pitch circles are equal to the intervals on the rack pitch line, are laid off. The intervals on the pitch lines of the conjugate gears are numbered in the reverse order to those on the pitch line of the original rack and represent the successive pitch points as the original rack moves along to the corresponding positions. To determine the profiles of the conjugate gear teeth, a series of similar triangles are constructed. From pitch point 9′ on the conjugate gear, an arc is drawn equal in radius to the distance from the pitch point to contact point 9 (shown at *A*). From the center of the conjugate gear, another arc is drawn, equal in radius to the distance from the corresponding center at *A* to the contact point 9. The intersection of these two arcs gives a point on the profile of the conjugate gear. Other points of the conjugate gear-tooth profiles are established in the same manner. At *B* and *C*, in Fig. 7, are the two conjugate gear-tooth forms that will run with each other and also with the basic

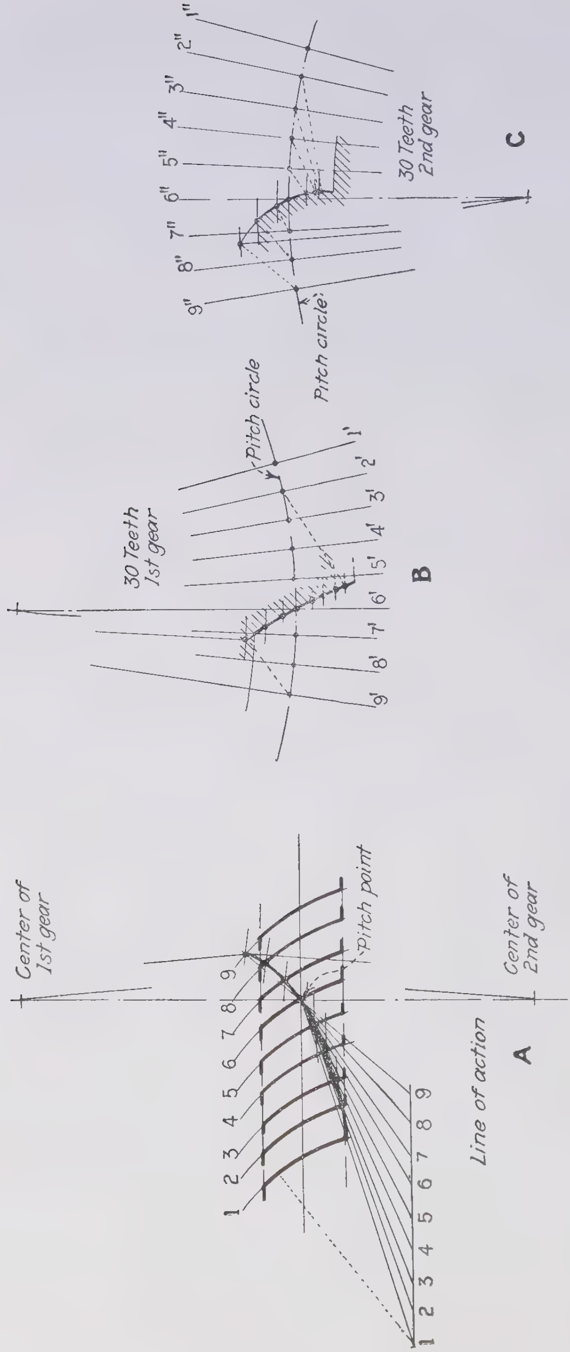

FIG. 7.—Construction of conjugate tooth forms from a basic rack with circular-arc profiles.

rack: the profile at B mates with the outside surface of the rack as drawn, and the profile at C mates with the inside surface or the reverse of the rack, as drawn in Fig. 6

PROBLEM 5.—*Given an arbitrary line of action, to construct the basic-rack form.*

This is a problem difficult to solve. The following statement appears in Grant's "Treatise on Gearing."

> When the line of action alone is given, the odontoids (conjugate gear-tooth profiles) for given pitch lines are fully determined, but there seems to be no simple graphical method of constructing them except for special cases.

The following method can be used, however, to get an extremely close approximation to the conjugate profiles. The larger the scale of the drawing, and the closer the intervals that are used, the closer the approximation will be. As a matter of fact, it is possible to carry this approximation to the point where errors of execution will introduce much greater errors than those of the method itself. This method is based on the assumption of uniform curvature of the profile of the basic rack between two successive intervals. The amount of error introduced by this method, therefore, depends upon the amount of change of curvature in the profile of the basic rack between two successive intervals, a change which is usually very small.

In Fig. 8 is shown an arbitrary line of action with a given pitch line. The first step is to lay off along the pitch line equal intervals, which are numbered for identification. The next step is to draw semicircles through each interval and the pitch point. It is not necessary to draw the complete semicircle, as an arc of it which intersects the line of action and the pitch line is sufficient. A line drawn from the pitch point to the intersection of the semicircle and the line of action will be a normal to the basic-rack form. A line connecting the intersections of the semicircle with the line of action and the pitch line will be perpendicular to the normal, as this line and the normal are inscribed in a semicircle, and will therefore be tangent to the basic-rack profile when moved to that specific position. Such tangents are shown in Fig. 8, numbered for identification.

The next step is to build up the basic-rack profile from these tangents. Intervals are established halfway between the original intervals and lettered for identification. These are

arcs of semicircles through the pitch point and the intermediate intervals on the pitch lines. Lines parallel to the pitch line are next drawn through the intersections of these arcs with the line of action and lettered to correspond. We are now ready to build up the profile of the basic rack from the tangents.

Through the pitch point a line is drawn between the parallel lines *f* and *g*, this line being perpendicular to the line of action at the pitch point. This gives the first section of the profile of the basic rack. From the intersection of this line with *ff'*, another line is drawn to *ee'*, parallel to the tangent 6. This gives the

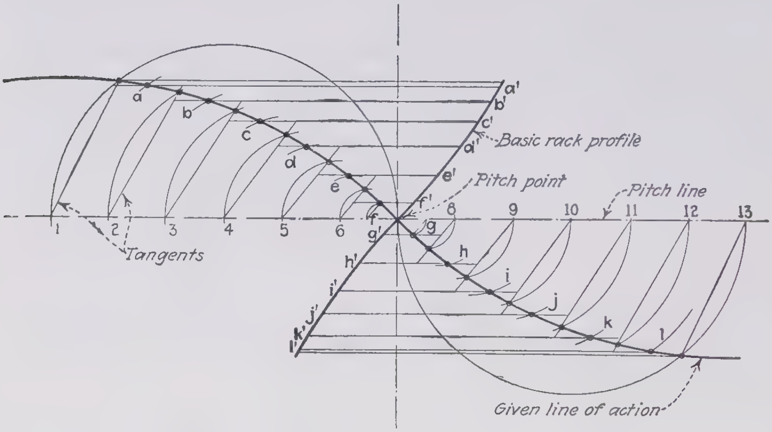

Fig. 8.—Basic-rack profile derived from a given line of action.

second section of the basic-rack profile. The same process is repeated until the entire profile of the basic is completed. A study of Fig. 8 should make the method clear. This method gives the profile as a series of straight lines. A smooth, curved line would be drawn, tangent to all these straight-line sections, which would represent the profile of the basic rack.

Once the basic-rack profile is established, the forms for conjugate gear teeth could be constructed in the same manner as illustrated in Fig. 7.

Limitations to Conjugate Action.—When the shape of the line of action is such that a tangent circle can be drawn to it with the center of the gear as its center, conjugate tooth action must cease at the point of tangency to avoid interference. Conjugate action can take place only along the line of action. Referring to *A*, in Fig. 9, if the diameter of the gear is carried beyond the

tangent circle a to the dotted line, the portion of the tooth beyond the tangent circle cannot come into proper action, because correct contact cannot exist away from the line of action. The same condition exists at B, in Fig. 9. Correct contact cannot take place below the tangent circle c. In order to avoid interference, the outside of the mating gear must not extend beyond the radius b, which is drawn from the center of the mating gear to the point of tangency of the circle c with the line of action.

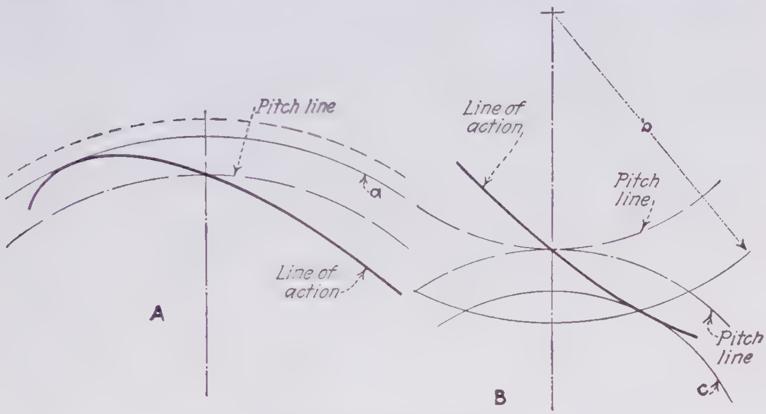

Fig. 9.—Limits of conjugate tooth action.

Requirements for Interchangeable Tooth Forms.—No consideration to the subject of interchangeability was given in the preceding examples. For example, the gear-tooth form shown at B, in Fig. 7, will mate properly with the gear-tooth form shown at C; but two gears with the same tooth form as shown at C will not run together properly. In order to obtain this interchangeability between gears, their line of action must be symmetrical in relation to the pitch point. When this condition is met, the profile of the basic rack is symmetrical about its pitch line, and all gears of all numbers of teeth that are conjugate to such a basic rack will also be conjugate to each other. In Fig. 8, the line of action is symmetrical in relation to the pitch point, and the profile of the resulting basic rack is symmetrical in relation to the pitch line. Thus, all gears which are conjugate to this basic rack will also be conjugate to each other.

One of the first regular curves used for gear-tooth profiles was the cycloidal form. Theoretically, it has many points of advantage. The practical difficulties of accurately producing

it, however, are responsible for its retirement from the field of commercial gears.

Wilfred Lewis has said:

> The practical consideration of cost demands the formation of gear teeth upon some interchangeable system. The cycloidal system cannot compete with the involute, because its cutters are formed with greater difficulty and with less accuracy, and a further expense is entailed by the necessity for more accurate center distances. Cycloidal teeth must not only be accurately spaced and shaped but their wheel centers must be fixed with equal care to obtain satisfactory results.

George B. Grant wrote in his excellent "Treatise on Gearing."

> There is no more need of two different kinds of tooth curves for gears of the same pitch than there is need for two different threads for standard screws, or two different coins of the same value, and the cycloidal tooth would never be missed, if it were dropped altogether. But it was first in the field, is simple in theory, is easily drawn, has the recommendation of many well-meaning teachers, and holds its position by means of "human inertia," or the natural reluctance of the average human mind to adopt a change, particularly a change for the better.

Although cycloidal forms are seldom used today for gear-tooth profiles, they are widely used for impellors of pressure blowers and for other special applications. Their action in these cases is conjugate gear-tooth action, although the actual rotation of the shafts is usually controlled by other gears. A brief analysis of these cycloidal forms seems, therefore, to be in order.

THE CYCLOID

The path described by a point on the circumference of a circle which rolls upon a straight line is called a *cycloid.* When the point where the curve meets the straight line is the origin, the equation of this curve and its derivation is as follows:

Referring to Fig. 10, let

a = radius of rolling circle
ψ = angle of rotation of rolling circle

The distance from the origin to the point of contact of the rolling circle with the straight line is equal to the length of the arc ψ at radius a, which equals $a\psi$. The generating point on the rolling circle is at a distance of $a \sin \psi$ from the vertical center line of the rolling circle, whence,

$$x = a(\psi - \sin \psi) \tag{1}$$

The generating point on the rolling circle is at a distance of $a \cos \psi$ below the center of this circle, whence,

$$y = a(1 - \cos \psi) \tag{2}$$

These two equations are the simplest form in which the equation of a cycloid can be given. They can be combined into a

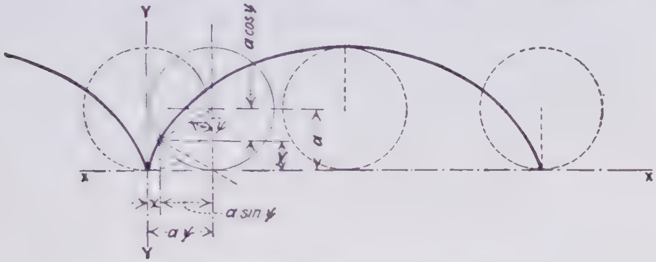

FIG. 10.—Development of the cycloid curve.

single equation, and the third variable ψ can be eliminated, as follows:

Transposing Eq. (2), we have

$$\cos \psi = \frac{a - y}{a}$$

$$\sin \psi = \sqrt{1 - \cos^2 \psi} = \frac{\sqrt{2ay - y^2}}{a}$$

$$\psi = \cos^{-1}\left(\frac{a - y}{a}\right)$$

Substituting in Eq. (1), we have

$$x = a \cos^{-1}\left(\frac{a - y}{a}\right) - \sqrt{2ay - y^2} \tag{3}$$

In general, Eqs. (1) and (2) will be found more convenient than Eq. (3). From Eqs. (1) and (2), we derive the equation for the tangent, as follows:

$$\tan \phi = \frac{dy}{dx} = \frac{\sin \psi d\psi}{(1 - \cos \psi)d\psi} = \frac{\sin \psi}{1 - \cos \psi} \tag{4}$$

where ϕ is the angle that the tangent makes with the X-axis.

THE EPICYCLOID

When a circle, tangent to a fixed circle externally, rolls upon it, the path described by a point on the circumference of the rolling circle is called an *epicycloid*. Taking the origin at the center of

the fixed circle and the Y-axis passing through the point where the curve meets the fixed circle, we derive its equation as follows:

Referring to Fig. 11, let

a = radius of fixed circle
b = radius of rolling circle
ψ = angular movement of rolling circle on fixed circle
β = angle of rotation of rolling circle

The length of the arc on the fixed circle from the point of contact between the two circles and the Y-axis is equal to $a\psi$.

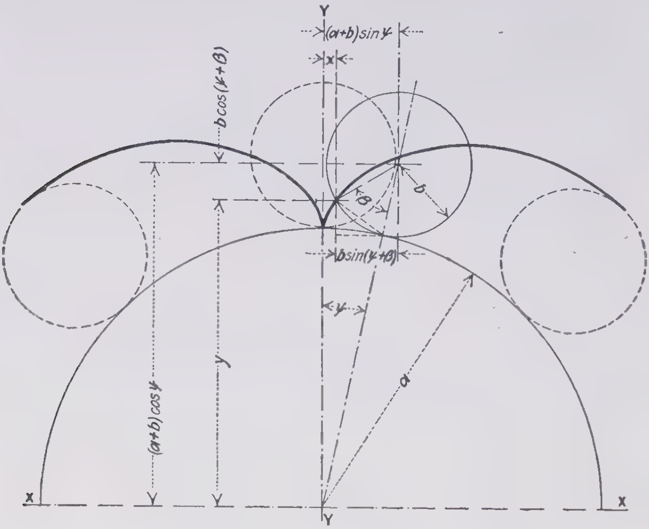

FIG. 11.—Development of the epicycloid curve.

This arc is equal to the length of the arc on the rolling circle from the generating point to the point of contact between the two circles, which, in turn, is equal to $b\beta$, whence,

$$a\psi = b\beta \qquad \beta = \frac{a}{b}\psi$$

From Fig. 11, we get

$$x = (a + b) \sin \psi - b \sin (\psi + \beta) =$$
$$(a + b) \sin \psi - b \sin \left(\frac{+ b}{b}\right)\psi \quad (5)$$
$$y = (a + b) \cos \psi - b \cos (\psi + \beta) =$$
$$(a + b) \cos \psi - b \cos \left(\frac{a + b}{b}\right)\psi \quad (6)$$

These two equations are the simplest form in which the equation of the epicycloid can be given. They can be combined, and the third variable ψ eliminated, but this combined form would be too complex to serve any useful purpose here.

From Eqs. (5) and (6), we derive the equation for the tangent, as follows:

$$\tan \phi = \frac{dy}{dx} = \frac{(a+b)\left[\sin\left(\frac{a+b}{b}\right)\psi - \sin\psi\right]d\psi}{(a+b)\left[\cos\psi - \cos\left(\frac{a+b}{b}\right)\psi\right]d\psi} = \frac{\sin\left(\frac{a+b}{b}\right)\psi - \sin\psi}{\cos\psi - \cos\left(\frac{a+b}{b}\right)\psi} \qquad (7)$$

where ϕ is the angle that the tangent makes with the X-axis.

THE HYPOCYCLOID

When a circle, tangent to a fixed circle internally, rolls upon it, the path described by a point on the circumference of the rolling

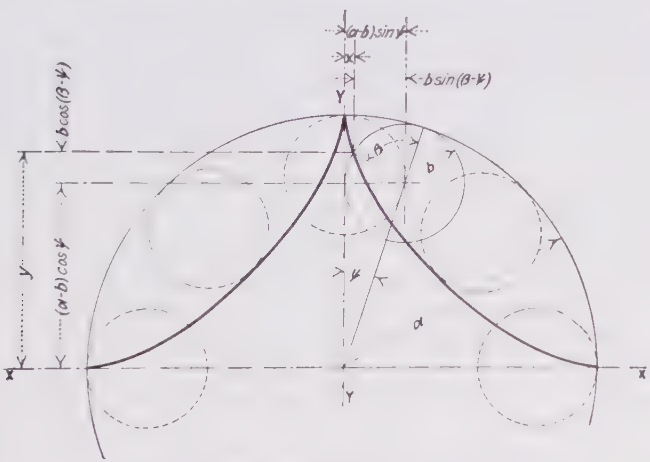

FIG. 12.—Development of the hypocycloid curve.

circle is called a *hypocycloid.* Taking the origin at the center of the fixed circle and the Y-axis passing through the point where the curve meets the fixed circle, we derive its equation as follows:

Referring to Fig. 12, let

a = radius of fixed circle
b = radius of rolling circle
ψ = angular movement of rolling circle on fixed circle
β = angle of rotation of rolling circle

$$a\psi = b\beta \qquad\qquad \beta = \frac{a}{b}\psi$$

From Fig. 12, we get

$$x = (a - b)\sin\psi - b\sin(\beta - \psi) = (a - b)\sin\psi - b\sin\left(\frac{a - b}{b}\right)\psi \tag{8}$$

$$y = (a - b)\cos\psi + b\cos(\beta - \psi) = (a - b)\cos\psi + b\cos\left(\frac{a - b}{b}\right)\psi \tag{9}$$

These two equations are the simplest form in which the equation of a hypocycloid can be given. From them we get, for the tangent,

$$\tan\phi = \frac{dy}{dx} = \frac{(a - b)\left[\sin\left(\frac{(a - b)}{b}\right)\psi - \sin\psi\right]d\psi}{(a - b)\left[\cos\psi - \cos\left(\frac{a - b}{b}\right)\psi\right]d\psi} = \frac{\sin\left(\frac{a - b}{b}\right)\psi - \sin\psi}{\cos\psi - \cos\left(\frac{a - b}{b}\right)\psi} \tag{10}$$

where ϕ is the angle that the tangent makes with the X-axis.

Line of Action.—The normals to the cycloidal profiles are perpendicular to the tangents. In Fig. 10, a dotted line is shown from the generating point to the point of contact of the rolling circle with the straight line. The lengths of the legs of the right triangle of which this dotted line is the hypotenuse are, respectively

$$a\sin\psi \text{ and } a(1 - \cos\psi)$$

The value of the cotangent of the angle between this dotted line and the X-axis is

$$\frac{a\sin\psi}{a(1 - \cos\psi)} = \frac{\sin\psi}{1 - \cos\psi}$$

A comparison of this equation with the one for the tangent of the cycloid, Eq. (4), shows equal values. This dotted line is therefore perpendicular to the tangent. In other words, this dotted line is the normal to the cycloid. Thus, the normal to the cycloid passes through the generating point and the point of contact of the rolling circle with the straight line upon which it rolls.

In Fig. 11, a dotted line is shown between the generating point and the point of contact between the fixed and the rolling circles. Another dotted line is shown from this point of contact parallel to the X-axis. The lengths of the legs of the right triangle thus formed are, respectively,

$$b\left[\cos\psi - \cos\left(\frac{a+b}{b}\right)\psi\right] \text{ and } b\left[\sin\left(\frac{a+b}{b}\right)\psi - \sin\psi\right]$$

The value of the cotangent of the angle included between the two dotted lines is

$$\frac{b\left[\sin\left(\frac{a+b}{b}\right)\psi - \sin\psi\right]}{b\left[\cos\psi - \cos\left(\frac{a+b}{b}\right)\psi\right]} = \frac{\sin\left(\frac{a+b}{b}\right)\psi - \sin\psi}{\cos\psi - \cos\left(\frac{a+b}{b}\psi\right)} \tag{11}$$

A comparison of this value with the equation for the tangent of the epicycloid, Eq. (7), will show that they are identical. *The normal to the epicycloid, therefore, passes through the generating point and the point of contact of the fixed and rolling circles.*

In like manner, it will be found that the normal to the hypocycloid also passes through the generating point and point of contact of the fixed and rolling circles.

Application to Tooth Forms.—This point of contact of the rolling circle with the line upon which it rolls represents the pitch point when these curves are used as gear-tooth profiles. The generating point represents a point of contact between mating cycloidal curves. In all cases of mating cycloids, the size of the rolling circles must be identical. The line of action between such forms is, therefore, the outline of the rolling circles, as shown in Fig. 13, and the fixed circles or the straight line upon which the rolling circles roll to generate the cycloidal forms become the pitch circles or the pitch line of the gears or racks, respectively.

When the size of the rolling circle is made one-half the size of the fixed circle, the equation of the hypocycloid becomes as follows:

$$a = 2b$$

Equation (8) becomes

$$x = \frac{a}{2}(\sin \psi - \sin \psi) = 0$$

Equation (9) becomes

$$y = \frac{a}{2}(\cos \psi + \cos \psi) = a \cos \psi$$

The shape of this cycloid is a radial line along the Y-axis. Thus, to design a gear form with radial flanks below the pitch

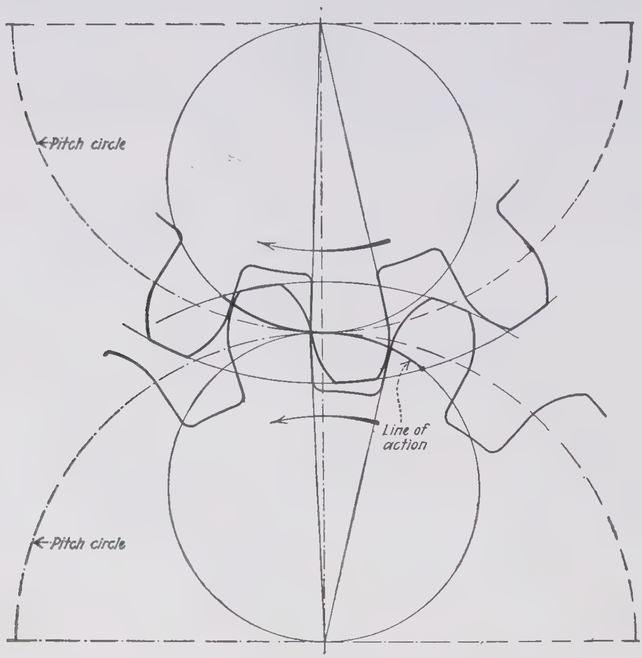

FIG. 13.—Line of action between mating cycloids.

circle, the diameter of the rolling circle is made one-half the diameter of the pitch circle.

Design of Cycloidal Rotors for Blowers.—The cycloidal form is extremely well adapted for use in blowers and is widely used in those known as the "Root" type. As a matter of fact, it would also be a more efficient form than the involute now used in oil and

water pumps, but it would require a more elaborate construction to realize this improved efficiency. The action between the two rotors is conjugate gear-tooth action, but the pressure angle between them rises so high that one will not drive the other through the whole cycle, so that other gears must be used to drive them when of spur gear type.

In Fig. 14 is shown a pair of two-lobed rotors of full cycloidal form. In this case, the diameter of the rolling circle is one-quarter the size of the fixed or pitch circle. If three lobes were desired, the diameter of the rolling circle would be one-sixth the diameter of the pitch circle.

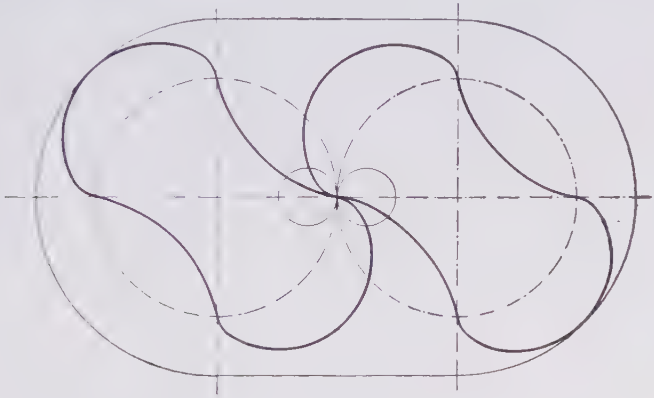

FIG. 14.—Blower with a pair of two-lobed rotors of full-cycloidal form.

One of the disadvantages of the cycloidal form is the practical difficulty of producing it. If the rotors are small enough to be generated on a gear shaper with a pinion-shaped generating tool, the form of such a tool can be readily generated by taking advantage of the fact that when the size of the rolling circle is one-half that of the pitch circle, the form of the resulting hypocycloid is a straight line.

As an example, we will assume that the pitch circle of the rotors shown in Fig. 14 is 4 inches in diameter. The diameter of the rolling circle is one-quarter of this, or 1 inch. By making a cutter 2 inches in diameter, of semicircular form, we can generate the corresponding epicycloid on a similar blank, as indicated in Fig. 15. This epicycloid, together with the straight line through the center, gives us the complete form of a one-lobed cutter which would generate the desired form on the rotors. This method

was originated by Paul M. Mueller, of the Pratt and Whitney Company.

This form of cutter, however, would be impractical if the rotors were of any appreciable length, as the form extends through the center of the cutter, and the present design of such gear-generating machines does not permit the excessive height above the work

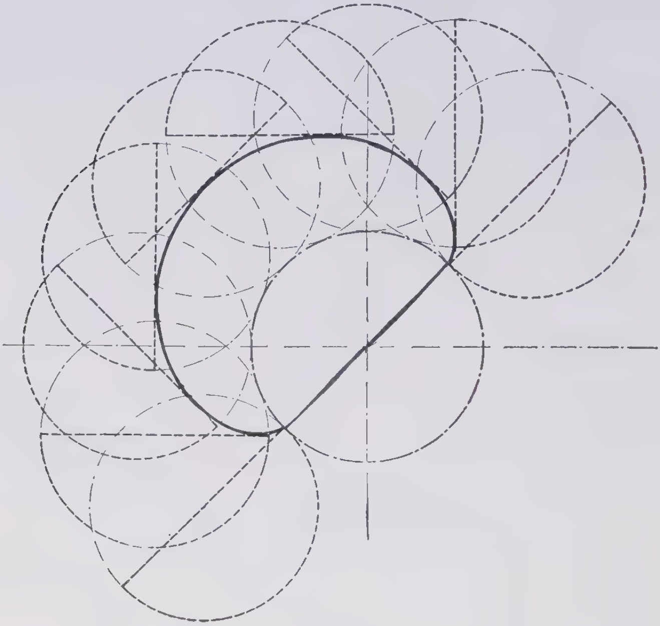

Fig. 15.—Straight-line hypocycloid generating the corresponding epicycloid.

table that would be necessary. A template could be produced, however, from which form-milling cutters could be made or which could be used as a template for shaping or which would represent the form of a two-lobe pinion-type cutter. This last could be used directly on the gear shaper, so long as the length of the rotors is not greater than the maximum face of gear which the machine can produce.

THE SEGMENTAL FORM

For generating machines that use a cutter of basic-rack form, the segmental form is a more practical and better tooth form to use than the cycloidal. This form is a close approximation to the cycloid and is composed of arcs of circles, whose centers are off

the pitch line, as shown in Fig. 16. The action between rotors generated from such a basic rack would be theoretically correct. The line of action, instead of being two perfect circles, is straightened out near the pitch line and flattened off at the top and bottom of the two lobes. This is also shown in Fig. 16. As this form

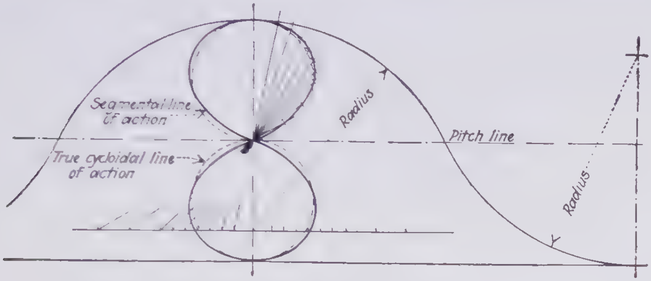

FIG. 16.—Rack-type cutter with circular-arc profiles.

is composed of arcs of circles, templates or form tools can be very accurately made on master plates, if desired. Due to the fact that its basic-rack form can be more accurately reproduced than the cycloid, the segmental form may be used to advantage, in many cases, in place of the cycloid.

SPLINE SHAFTS

The problems of determining the forms of hobs to produce spline shafts, ratchets, sprocket wheels, etc., are essentially gear problems. As an example, we will determine the form of the basic rack to produce the usual straight-sided spline shaft shown in Fig. 17.

The first step is to establish the pitch line which will be employed. In this example, we will take it as the outside diameter of the spline shaft.

The next step is to establish the line of action. This is necessary in order to make sure that with the established pitch line, complete conjugate action exists. If it should be possible to draw tangent circles from the center of the spline shaft to this line of action, and if these tangent circles should lie inside of the profile of the spline or other form, complete generation could not take place (see Fig. 9). If this condition should exist, it would be necessary to establish a different pitch line which would avoid this condition. The method of determining the line of action is illustrated in Fig. 3.

The profile of the basic rack is next established. This is accomplished by the same method as was illustrated in Fig. 4. The form of the basic rack for the spline shaft has been developed in Fig. 17. The line of action is first established, and from it the rack form. This construction is laid out to double size at the top of the illustration.

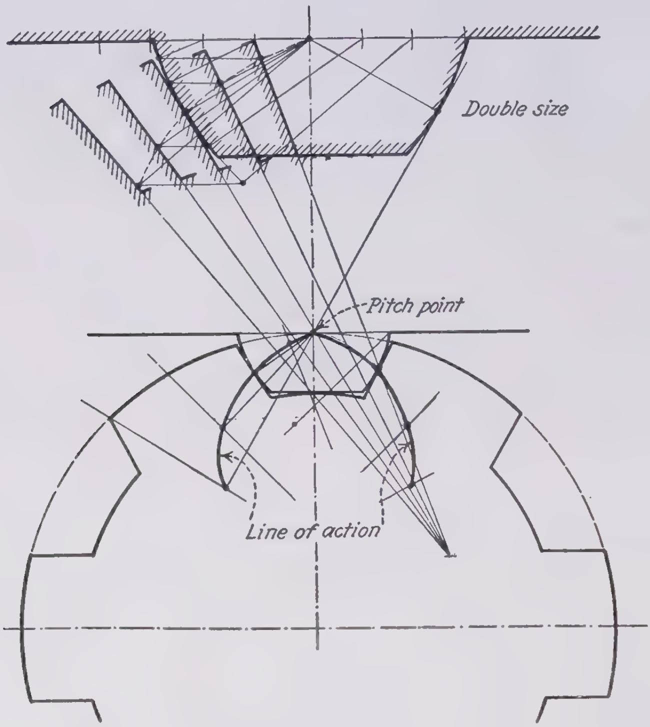

Fig. 17.—Development of the basic rack for a spline shaft.

The form thus determined is used as the shape of the hob tooth which will be employed to produce the desired form. Theoretically, a correction in this form is required because of the side-cutting action of the hob in operation. In the majority of cases, however, the error introduced by the side-cutting action of the hob is less than the probable error in actually producing the form on the hob, so that this further refinement is seldom required.

CHAPTER II

THE INVOLUTE CURVE AND ITS PROPERTIES

The application of cycloid curves to gear-tooth forms is rather limited when compared with the use of the involute curve. At the present time, this curve is almost exclusively used for gear-tooth profiles. It meets all the requirements for a gear-tooth profile and, in addition, has so many other valuable properties that it stands in a class by itself. These properties free it from many of the restrictions of other gear-tooth shapes. In order to appreciate its many valuable features, it is best to study it by itself rather than as one of a group of gear-tooth forms.

The involute is the curve that is described by the end of a line which is unwound from the circumference of a circle, as illustrated in Fig. 18. The circle from which the line is unwound is commonly known as the "base circle." The equation of the involute and its derivation is as follows:

Where a = radius of base circle
b = length of generating line
r = length of radius vector
θ = vectorial angle
α = angle between radius vector and radial line of base circle to point of tangency of generating line with base circle

It is most convenient to use the circular measure of some of the angles when dealing with the involute curve. The circular measure of an angle is a radian, which is the angle subtended by an arc whose length is equal to the radius. The circular measure of a complete circle, or 360 deg., is, therefore, 2π radians. In all of the following equations where the symbol of an angle is used without any expressed trigonometric function, the angle is expressed in radians.

In Fig. 18, the length of the generating line is equal to the length of the arc which subtends the two angles θ and α, because

this line has been unwound from that portion of the circumference of the base circle. . Thus, we have

$$b = a(\theta + \alpha)$$

But b is also the leg of a right triangle, whence,

$$b = a \tan \alpha$$

Combining these two equations,

$$a(\theta + \alpha) = a \tan \alpha$$
$$\theta + \alpha = \tan \alpha$$
$$\theta = \tan \alpha - \alpha \qquad (12)$$

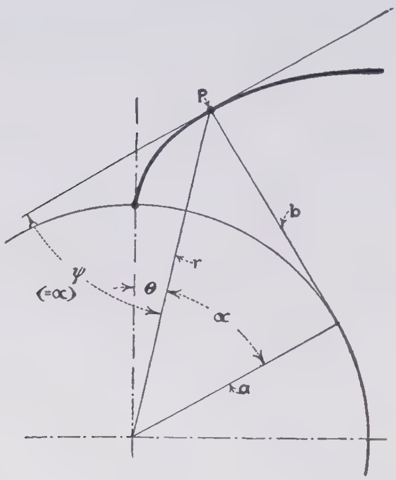

FIG. 18.—Generation of the involute curve.

From the same right triangle, we have

$$r = \frac{a}{\cos \alpha} \qquad (13)$$

These last two equations are the simplest form in which the equations of the involute of a circle can be given. They can be combined, and the third variable α eliminated, as follows: Referring to Fig. 18,

$$\tan \alpha = \frac{b}{a}$$

$$b = \sqrt{r^2 - a^2}$$

whence,

$$\tan \alpha = \frac{\sqrt{r^2 - a^2}}{a} = \sqrt{\left(\frac{r}{a}\right)^2 - 1}$$

or

$$\alpha = \tan^{-1}\sqrt{\left(\frac{r}{a}\right)^2 - 1}$$

Substituting these values in Eq. (12), we have

$$\theta = \sqrt{\left(\frac{r}{a}\right)^2 - 1} - \tan^{-1}\sqrt{\left(\frac{r}{a}\right)^2 - 1} \tag{14}$$

Equation (14) is the polar equation of the involute curve. We know that the equation of the tangent to a curve measured from the radius vector is

$$\tan\psi = r\,\frac{d\theta}{dr}$$

$$\frac{d\theta}{dr} = \frac{\frac{r}{a^2}}{\sqrt{\left(\frac{r}{a}\right)^2 - 1}} - \frac{\frac{r}{a^2}}{\frac{r^2}{a^2}\sqrt{\left(\frac{r}{a}\right)^2 - 1}} = \left(\frac{r}{a^2}\right)\frac{\left(\frac{r}{a}\right)^2 - 1}{\frac{r^2}{a^2}\sqrt{\left(\frac{r}{a}\right)^2 - 1}} = \frac{\sqrt{\left(\frac{r}{a}\right)^2 - 1}}{r}$$

$$r\frac{d\theta}{dr} = \tan\psi = \sqrt{\left(\frac{r}{a}\right)^2 - 1} = \tan\alpha \tag{15}$$

hence,

$$\psi = \alpha$$

The tangent to the involute is, therefore, perpendicular to the generating line, as shown in Fig. 18, or conversely, *the generating line is the normal to the involute curve.*

The radius of curvature at any point is equal to the length of the generating line from that point to its point of tangency to the base circle. In other words, the radius of curvature at any point is equal to b, or $\sqrt{r^2 - a^2}$.

A simple conception of the involute curve is that of a uniform-rise cam, where the rise per revolution along a line tangent to a base circle of radius a is equal to the circumference of the circle, or $2\pi a$. This is shown in Fig. 19. If this cam is revolving at a uniform rate of speed in the direction indicated by the arrow, the cam roll will rise at a uniform rate. If the cam revolves in the reverse direction, the roll will fall accordingly.

The line which is tangent to the base circle is the line of action or the path of contact. It will be noted that *this line of action is a straight line* and is therefore symmetrical about any point in this line. This is one of the unique properties of the involute curve which sets it apart from all other gear-tooth curves.

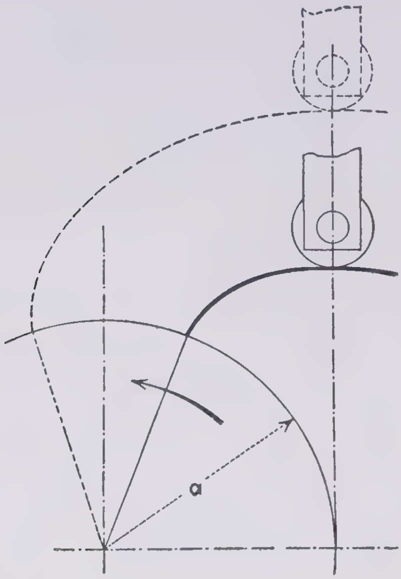

FIG. 19.—The involute gives a uniform cam rise.

Action of One Involute against Another.—If, instead of acting against a cam roll, the involute acts against another involute, we have the conditions shown in Fig. 20. The point of contact between two involutes is that point where the tangents to the two curves coincide. The tangents to both involutes are always perpendicular to their generating lines. The tangents to two involutes in contact coincide only when the generating line of one involute is a continuation of the generating line of the second involute. Therefore, *the locus of the points of contact between two involutes is the common tangent to the two base circles*, as shown in Fig. 20.

When one involute is revolved at a uniform rate of motion, the length of the line of action b_1 from its point of tangency to the base circle a_1 to the involute profile at P changes uniformly. If the direction of the rotation is in the direction shown by the

arrow in Fig. 20, the length of this line increases. At the same time, the length of the line of action b_2 from the point P to its point of tangency with the base circle a_2 is shortened at a correspondingly uniform rate, because the total length of the common tangent remains constant. This means that the second involute must revolve at a *uniform* rate in the direction shown by the arrow in Fig. 20.

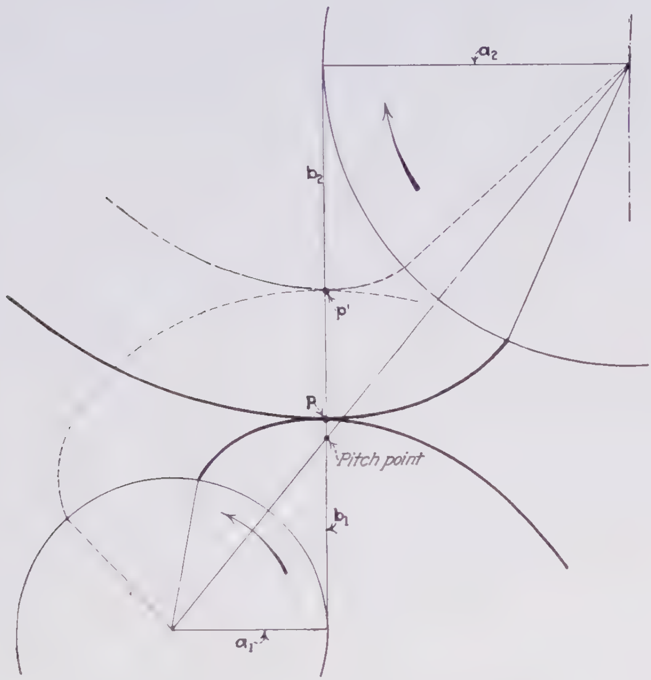

FIG. 20.—Action of two involutes in contact.

Base Circles Determine Speed Ratio.—*The relative rate of motion depends only upon the relative sizes of the two base circles.* No matter what the distance is between the centers of the two base circles, when one involute acts against another, contact between them occurs only along the common tangent to the two base circles, and their relative rates of revolution remain the same. If the two base circles are identical in size, this rate is also identical for both. If one base circle is double the size of the other, the rate of revolution of the larger involute is one-half

that of the smaller. This is because the larger base circle revolves only half as far as the smaller to wind up the length along the line of action that the smaller one has unwound. The conditions are exactly the same as though two pulleys were set up connected by a crossed belt. Thus, *the relative rates of revolution of two involutes, which act against each other, are in inverse proportion to the sizes of their base circles.*

When S_1 = rate of revolution of first involute
S_2 = rate of revolution of second involute
a_1 = radius of base circle of first involute
a_2 = radius of base circle of second involute

$$\frac{S_1}{S_2} = \frac{a_2}{a_1}$$

The relative rates of the two involutes may be represented by two plain disks which drive each other by friction. Such disks are commonly known as "pitch disks," while their diameters are called "pitch diameters." *An involute has no pitch diameter until it is brought in contact with another involute.* This is another unique property of the involute curve which sets it apart from all other gear-tooth curves. All other gear-tooth curves must be developed from a definite pitch line. The involute has no fixed pitch line, but any diameter on it is a potential pitch diameter. *The form of the involute depends solely upon the size of its base circle.*

In Fig. 21, two involutes are shown in contact at different center distances. The common tangent to the two base circles is the line of action. We have seen before that the radii of the base circles are in inverse proportion to the rates of revolution of the involutes. The radii of two pitch disks which represent the same relative rates of revolution must be directly proportional to the radii of the base circles of their respective involutes.

To prove this, in Fig. 21 we have two similar right triangles in which R_1 is the hypotenuse of one, while R_2 is the hypotenuse of the other; and a_1 is a leg of one, and a_2 is the corresponding leg of the other.

The intersection of the common tangent to the two base circles with the common center-line establishes the radii of the two pitch circles.

The angle between the common tangent to the two base circles and a line perpendicular to their common center-line is called the *pressure angle*. This angle does not exist until two involutes are brought into contact. There is a definite relation between the pitch diameter and pressure angle of any given involute. Thus, for any established pitch diameter there is a corresponding pressure angle. It will be noted in Fig. 21, that although the same involutes are shown at A and B, they have different pitch circles

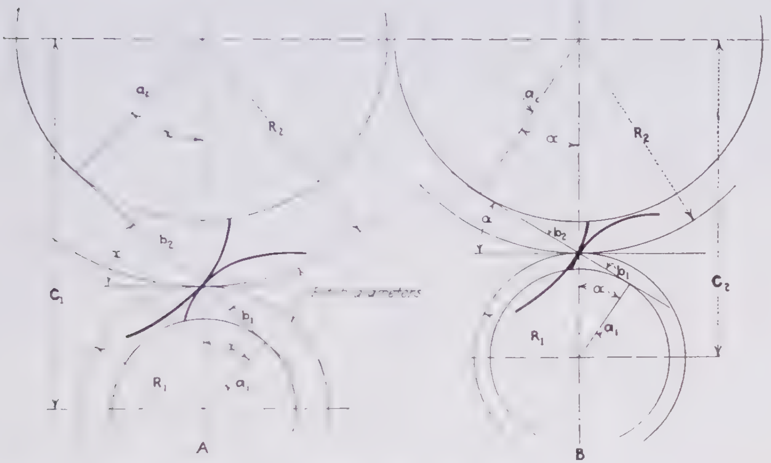

FIG. 21.—Influence of the base-circle center-line distance upon the line of action and the pitch diameter for mating involutes.

at A than at B, because the distance between the centers is different.

Thus, *both the pitch circles and the pressure angle of a pair of involutes depend solely upon the sizes of the base circles and the distance between their centers.*

When C = center distance

$$C = R_1 + R_2$$

$$R_1 = \frac{a_1 R_2}{a_2}$$

$$C = \frac{a_1 R_2}{a_2} + R_2 = R_2\left(\frac{a_1 + a_2}{a_2}\right)$$

whence,

$$R_2 = \frac{a_2 C}{a_1 + a_2} \tag{16}$$

In like manner,

$$R_1 = \frac{a_1 C}{a_1 + a_2} \tag{17}$$

Referring again to Fig. 21, when

α = pressure angle of the pair of involutes, we find

$$\cos \alpha = \frac{a_1 + a_2}{C} \tag{18}$$

Action of an Involute against a Straight Line.—When an involute acts against a straight line, we have the conditions shown in Fig. 22. The straight line is tangent to the involute and is always perpendicular to its line of action. When it is con-

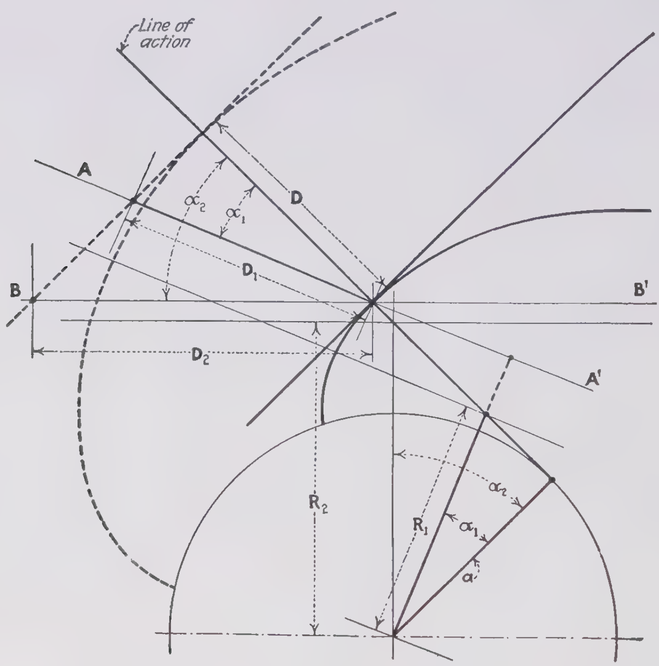

Fig. 22.—Action of an involute against a straight line.

strained to move only in the direction of the line of action, it will be moved at a corresponding and uniform rate, if the involute is revolved at a uniform rate. The distance this line moves depends upon the size of the base circle of the involute. Thus, for one complete revolution of the involute, the line will move

along the line of action a distance equal to the circumference of the base circle.

We will now consider the motion of this straight line when it is constrained so that it can move only in the direction of the line AA'. If we designate the distance which the line travels along the line AA' as D_1, the distance the line has moved along the line of action as D, and the angle between the line of action and the line AA' as α_1, we have the following relationship:

$$D_1 = \frac{D}{\cos \alpha_1}$$

As the value of D changes uniformly, and since the value of α_1 is constant, the value of D_1 changes uniformly. As $\cos \alpha_1$, can never be greater than unity, the value of D_1 will never be smaller than D. Therefore, when the line against which the involute acts is constrained so that it moves only in the direction of the line AA', the distance it travels along this line will be greater than the distance along the line of action, but its rate of motion will be uniform as long as the rate of rotation of the involute is uniform.

If the involute should make one complete revolution, the value of D would become $2\pi a$. The value of D_1 would then be $2\pi a/\cos \alpha_1$. This also represents the circumference of a pitch disc which runs with a straight edge parallel to the line AA'. The radius of this pitch circle thus becomes equal to $a/\cos \alpha_1$. In Fig. 22, the radius of this pitch circle is designated by R_1, and it is established by the intersection of the line of action with a radial line of the base circle which is perpendicular to the line AA'.

In like manner, when the line, against which the involute is acting, is constrained so that it moves only in the direction of the line BB', the distance R_2 becomes the radius of the pitch circle of the involute, and the angle α_2 becomes its pressure angle. The motion along the line BB' is also uniform when the rate of rotation of the involute is uniform. The distance D_2 equals $D/\cos \alpha_2$, and R_2 equals $a/\cos \alpha_2$.

SUMMARY OF INVOLUTE-CURVE PROPERTIES

It follows, then, that the involute curve has the following properties:

1. The shape of the involute is dependent only upon the size of the base circle.

2. If one involute, rotating at a uniform rate of motion, acts against another involute, it will transmit a uniform angular motion to the second, regardless of the distance between the centers of the two base circles.

3. The rate of motion transmitted from one involute to another depends only upon the relative sizes of the base circles of the two involutes. This rate of angular motion is in inverse proportion to the sizes of the two base circles.

4. The common tangent to the two base circles is the line of action. In other words, the two involutes will make contact with each other only along this common tangent of the two base circles.

5. The line of action of an involute is a straight line. Any point on this straight line may therefore be taken as a pitch point, and the line of action will remain symmetrical in regard to it.

6. The intersection of the common tangent to the two base circles with their common center-line establishes the pitch lines of mating involutes. No involute has a pitch line until it is brought into contact with another involute or a straight line constrained to move in a definite fixed direction.

7. The pitch diameters of two involutes acting together are directly proportional to the diameters of their base circles.

8. The pressure angle of two involutes acting together is the angle between the common tangent to the two base circles and a line perpendicular to their common center-line. No involute has a pressure angle until it is brought in contact with another involute or a straight line constrained to move in a fixed direction.

9. The pressure angle of an involute acting against a straight line constrained to move in a fixed direction is the angle between the line of action and a line representing the direction in which the straight line can move.

10. The pitch radius of an involute acting against a straight line constrained to move in a fixed direction is the length of a radial line, perpendicular to the direction in which the straight line can move, measured from the center of the base circle to its intersection with the line of action.

USE OF THE INVOLUTE FORM FOR GEAR TOOTH PROFILES

When the involute form is used as a gear-tooth profile, several involute curves are developed from the same base circle to form the profiles of the several teeth. As the gear-tooth profile is symmetrical, for the present we shall consider but one side of the teeth.

In Fig. 23 is shown the development of one side of the several teeth. Imagine a string with knots equally spaced wound about the circumference of the base circle. As this string is unwound, each knot will describe an involute curve. The distance between these involute curves, measured along any line tangent to the base circle, is always the same. This distance is equal to the length of the arc of the base circle between the origins of two successive involutes and is the *normal pitch* of the gear.

Thus, when Pn = normal pitch
a = radius of base circle
N = number of teeth in gear

$$Pn = \frac{2\pi a}{N} \tag{19}$$

In a pair of mating gears, the normal pitch must be identical in order to secure smooth, continuous action.

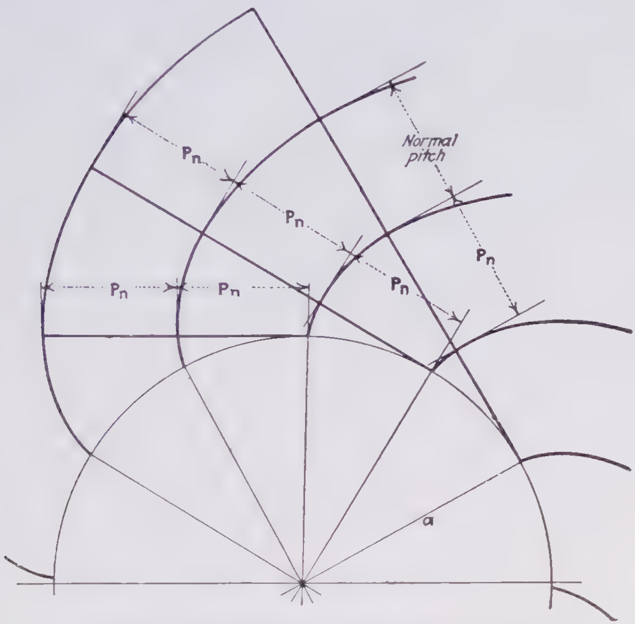

FIG. 23.—Development of successive tooth sides.

Duration of Contact.—One of the important factors in the design of gears which are to transmit power is that the involute profiles must be so selected that the second pair of mating teeth will be in contact before the first pair is out of contact. The proper amount of contact depends upon several conditions. If the form of the tooth is rugged, and the load, when applied toward the outer end of the tooth, causes practically no deflection, the amount of overlap may be small with satisfactory results.

The angle of action is the angle through which one tooth travels from the time it first makes contact with its mating tooth until it ceases to be in contact. The number of teeth in contact, or

the contact ratio, is the quotient of the angle of contact divided by the angle between two successive teeth on the gear. Thus, if an overlap of 0.6 exists, the number of teeth in contact is 1.6.

In Fig. 24, that part of the line of action which is intercepted by the two outside circles of a pair of mating gears, shown as a heavy line, is the length of the arc of the angle of action measured at the radius of the base circle. This length divided by the normal pitch, which is the length of an arc of the base circle between

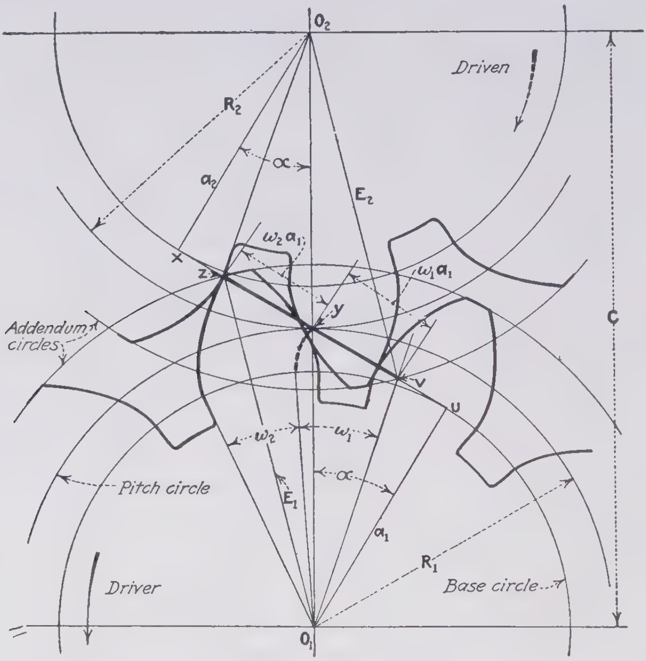

FIG. 24.—Arc and angle of action with involute profiles.

two successive teeth on the gear, gives the number of teeth in contact.

The angle of action is often divided into the angle of approach and the angle of recess. The angle of approach is the angle through which the tooth moves from the time it first comes into contact with its mating tooth until contact is made at the pitch point. The angle of recess is the angle through which the tooth moves from the time when the contact is at the pitch point until it ceases to be in contact with its mating tooth.

Referring to Fig. 24, let

α = pressure angle
ω_1 = angle of approach
ω_2 = angle of recess
N_1 = number of teeth in pinion
N_2 = number of teeth in gear
C = center distance
R_1 = radius of pitch circle of pinion
R_2 = radius of pitch circle of gear
E_1 = radius of addendum circle of pinion
E_2 = radius of addendum circle of gear,
P_h = normal pitch.

Simple formulas can be derived for the values of ω_1 and ω_2 and the number of teeth in contact by solving several right triangles. The angle of approach ω_1, in circular measure, is found by dividing the length of the line yv by the radius of the base circle a_1. This length is equal to the length xv minus the length xy.

Length of line

$$xy = R_2 \sin \alpha$$

Length of line

$$xv = \sqrt{(E_2)^2 - (a_2)^2}$$

whence,

$$\omega_1 = \frac{\sqrt{(E_2)^2 - (a_2)^2} - R_2 \sin \alpha}{a_1} \tag{20}$$

The angle of recess ω_2 is found in a similar manner by dividing the length of the line yz by a_1. This length is equal to the length zu minus the length yu.

Length of line

$$yu = R_1 \sin \alpha$$

Length of line

$$zu = \sqrt{(E_1)^2 - (a_1)^2}$$

$$\omega_2 = \frac{\sqrt{(E_1)^2 - (a_1)^2} - R_1 \sin \alpha}{a_1} \tag{21}$$

The number of teeth in contact is found by dividing the length of the line zv by the normal pitch Pn. The length of the line zv is equal to the sum of yz and yv.

$$zv = \sqrt{(E_2)^2 - (a_2)^2} + \sqrt{(E_1)^2 - (a_1)^2} - (R_1 + R_2) \sin \alpha$$

But

$$R_1 + R_2 = C$$

whence, number of teeth in contact =

$$\frac{\sqrt{(E_1)^2 - (a_1)^2} + \sqrt{(E_2)^2 - (a_2)^2} - C \sin \alpha}{Pn} \tag{22}$$

Sometimes the tooth form of a gear extends below the base circle. No involute action, however, can take place below the base circle. Thus, if the value of $\sqrt{(E_1)^2 - (a_1)^2}$ or $\sqrt{(E_2)^2 - (a_2)^2}$ is greater than $C \sin \alpha$, the total length of the line of action, the value of $C \sin \alpha$ must be substituted in place of the greater value, because the line of actual contact cannot extend beyond the point of tangency to either base circle.

Active Profile.—The active profile of a gear tooth is that portion of the tooth profile which actually comes in contact with its mating tooth along the line of action. In general, when the tooth design is such that excessive sliding exists, one or both active profiles will be short in proportion to the length of the whole tooth profile. When the amount of sliding is small, on the other hand, the active profiles will include most of the total tooth profile.

The extent of the active profile can be calculated by solving right triangles. Referring again to Fig. 24, the active profile of the pinion is equal to the difference in length between the outside radius of the pinion E_1 and the radial line o_1v. The line o_1v is the hypotenuse of a right triangle of which a_1 and the line uv are legs.

$$\text{Line } uv = C \sin \alpha - \sqrt{E_2^2 - a_2^2}$$

whence,

active profile of pinion =

$$\sqrt{a_1^2 + [C \sin \alpha - \sqrt{E_2^2 - a_2^2}]^2}$$

In similar manner,

active profile of gear =

$$\sqrt{a_2^2 + [C \sin \alpha - \sqrt{E_1^2 - a_1^2}]^2}$$

ROLLING AND SLIDING CONTACT

As pointed out before, the length of the generating line which is unwrapped from the base circle is the radius of curvature of the involute at any point. Figure 25 shows the position of this generating line at equal angular intervals. At the origin a the

length of the generating line is zero. At b it is infinitely longer. At c it is twice the length that it is at b. At d it is one and one-half times the length at c. At e it is one and one-third times the length at d, and so on. The radius of curvature thus increases rapidly in proportionate length near the base circle and more slowly as the curve departs farther from the base circle. In other words, the form near the base circle is very sensitive, that is, it has a small and rapidly changing radius of curvature,

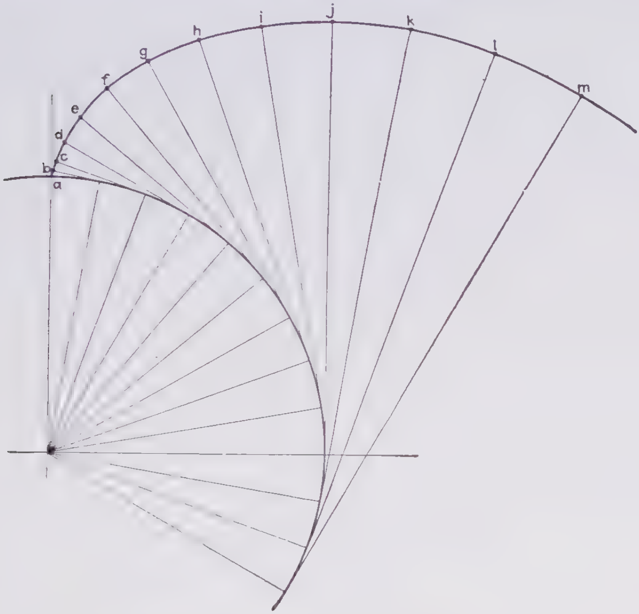

FIG. 25.—Variation of the radius of curvature of the involute.

but it becomes less sensitive the farther it departs from the base circle.

Sensitive curves of this type are most difficult to produce accurately, whether on gear-tooth forms or on other types of cams, and they should always be avoided wherever possible. Thus, *only in cases of necessity should the active profile of an involute gear extend to or very near the base circle.*

It will also be noted in Fig. 25 that the length of the curve ab is much less than the length bc; that bc is smaller than cd; etc. Thus, whether the involute is acting as a cam or is acting against another involute, the length of the curve that must pass through

the line of action for any series of equal angular movements changes constantly. The nearer the active part of the profile is to the base circle, the shorter is the length of the profile.

Thus, when two involutes are acting against each other, a combined rolling and sliding action takes place between them because of the varying lengths of equal angular increments on the profiles.

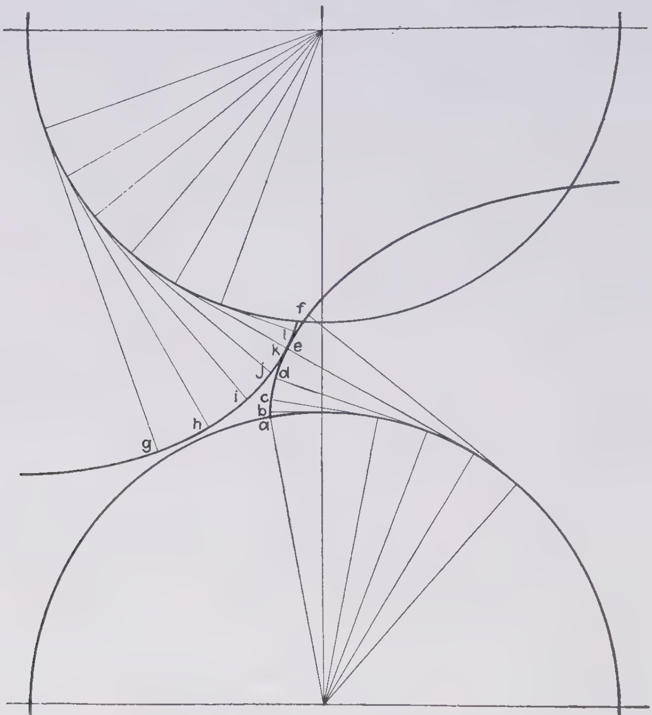

Fig. 26.—Sliding produced by unequal arcs of the involute.

In Fig. 26 are shown two equal involutes with the generating lines shown at equal angular intervals. The part of the profile *ab* on one involute comes into contact with the profile *gh* on the other. Profile *ab* is much nearer its base circle than *gh* and is therefore much shorter. The two profiles must slide against each other a distance equal to their difference in length to make up this difference. Profile *bc* is longer than *ab*, while profile *hi* is shorter than *gh*. The length *bc* is still much shorter than *hi*, but the amount of sliding will not be so great as with the previous sec-

tions of mating profiles. Spaces *cd* and *ij* are more nearly equal in length, *cd* being the shorter, so that still less sliding occurs. The profiles *de* and *jk* are almost equal in length, but the length of the profile *de* on the first involute is now slightly longer than its mating portion of the profile on the second involute. Thus, the slight amount of sliding that occurs now takes place in the opposite direction. The remaining sections of the profile of the first involute become increasingly longer, while those on the second involute become smaller, so that the amount of sliding increases again.

It is evident that the rate of sliding between two involutes acting against each other is constantly varying. The rate of sliding decreases to zero, changes its direction, and increases again. The actual amount of sliding is the same on both profiles, but it is distributed over different lengths of profiles, so that its rate or velocity is different.

This condition of sliding can be illustrated in a simple manner by considering the action between two disks of equal diameter. When these disks are rolled together at the same rate, so that each would make a complete revolution in the same time, pure rolling action occurs. This represents the action of two involutes in contact at the pitch point. When one of the disks is held stationary while the other revolves, sliding results. The amount of sliding is the same on both disks, but it is concentrated at one point on the stationary disk, while it is distributed over the whole surface of the revolving disk. The sliding action on the stationary disk represents the sliding action at the base circle of an involute gear tooth, while that on the revolving disk represents the sliding on that part of the addendum of the mating tooth which makes contact at the base circle. Again, when both disks are rolled together but each rotates at a different rate of speed, sliding develops. The amount of sliding is the same on both disks, but it is distributed over a larger part of the circumference of the more quickly moving disk than that of the slower one. The sliding action on the faster disk represents the sliding action on the addendum of an involute gear tooth, while that on the slower one represents the sliding on the dedendum of the tooth.

Formulas for computing the ratio of relative sliding, or the "specific sliding," as it is called, are derived as follows: The value of the specific sliding multiplied by the length of the profile will give the actual amount of sliding. In other words, the specific

sliding represents the quotient of the actual amount of sliding divided by the length of the profile on which this sliding occurs.

Referring to Fig. 27, let

N_1 = number of teeth in pinion
N_2 = number of teeth in gear
a_1 = radius of base circle of pinion
a_2 = radius of base circle of gear
b_1 = length of generating line of pinion to point of contact
b_2 = length of generating line of gear to point of contact
r_1 = radius on pinion to point of contact
r_2 = radius on gear to point of contact
C = center distance
α = pressure angle

We may consider any infinitely small portion of the involute profile as an arc of a circle with a radius equal to the length of the generating line to that point. The lengths of two such infinitely small arcs of the same angular magnitude will be proportional to the lengths of their respective radii. But the angular magnitudes of two such infinitely small arcs which form the point of contact of two involutes are inversely proportional to the diameters of their respective base circles, or to their respective number of teeth. If the ratio, therefore, of the lengths of the radii of the two infinitely small arcs in contact are directly proportional to the number of teeth in the two gears, pure rolling will result. If these lengths are not thus proportional, sliding will result. The rate of sliding will be equal to the difference between the relative lengths of these two infinitely small arcs divided by the relative

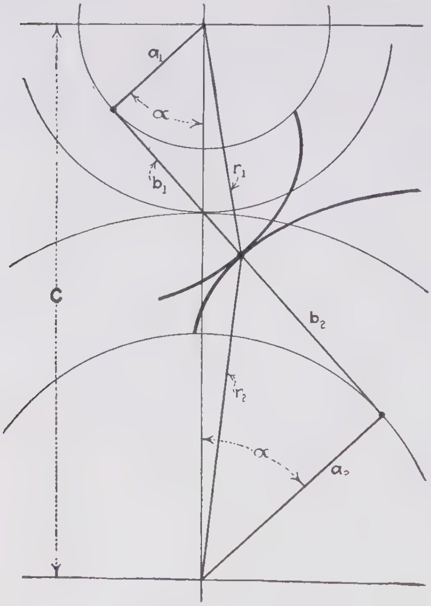

Fig. 27.—Relative sliding between two involutes.

length of one of them. Thus, we may write as the formula for specific sliding the following:

$$\text{Specific sliding on pinion} = \frac{b_1N_2 - b_2N_1}{b_1N_2} \tag{23}$$

$$\text{specific sliding on gear} = \frac{b_2N_1 - b_1N_2}{b_2N_1} \tag{24}$$

The total length of the line of action, or common tangent to the two base circles, equals $b_1 + b_2 = C \sin \alpha$

$$b_1 = \sqrt{(r_1)^2 - (a_1)^2} \tag{25}$$

and

$$b_2 = C \sin \alpha - b_1 \tag{26}$$

When the portion of the involute of the pinion at the base circle is in contact, we have the following:

$$b_1 = 0$$
$$r_1 = a_1$$
$$\text{Specific sliding} = \frac{0 - b_2N_1}{0} = -\infty$$

In like manner, the specific sliding at the base circle of the gear will be found equal to minus infinity.

When the portion of the involute of the pinion at the pitch circle is in contact, we have the following:

$$b_1 = a_1 \tan \alpha$$
$$b_2 = a_2 \tan \alpha$$
$$\text{Specific sliding} = \frac{a_1N_2 \tan \alpha - a_2N_1 \tan \alpha}{a_1N_2 \tan \alpha} = \frac{a_1N_2 - a_2N_1}{a_1N_2}$$

but

$$a_1 : a_2 = N_1 : N_2$$

whence,

$$a_1N_2 = a_2N_1$$

Substituting, we have

$$\text{specific sliding} = \frac{0}{a_1N_2} = 0$$

Thus, *pure rolling action occurs on the pitch line of involute gears.*

Determination of Sliding Velocity.—The actual sliding velocity at any point on the pinion may be determined by multiplying the

specific sliding by the actual velocity of the given point through the mesh or contact point. This velocity is equal to

$$\frac{b_1 V \cos \alpha}{a_1}, \text{ where } V = \textit{pitch-line velocity}$$

Whence,

$$\text{sliding velocity on pinion} = \frac{b_1 V \cos \alpha}{a_1}\left(\frac{b_1 N_2 - b_2 N_1}{b_1 N_2}\right)$$
$$= V \cos \alpha \left(\frac{b_1 N_2 - b_2 N_1}{a_1 N_2}\right) \quad (27)$$

Similarly, the actual sliding velocity at any point on the gear becomes as follows:

$$\text{Sliding velocity on gear} = V \cos \alpha \left(\frac{b_2 N_1 - b_1 N_2}{a_2 N_1}\right) \quad (28)$$

The actual sliding velocities are the same at any given point of contact for both the gear and the pinion. The foregoing equations will give the same numerical results for both the gear and the pinion but with different signs, which indicate the difference in the nature of the sliding.

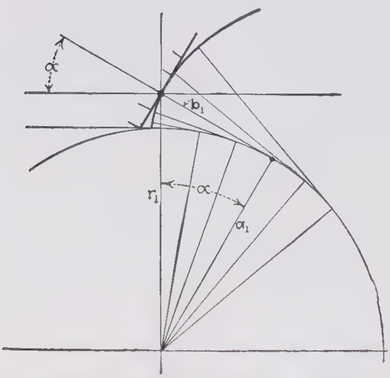

Fig. 28.—Relative sliding between an involute and its rack.

A rack may be considered as an involute curve with an infinitely large base circle. The lengths of the portions of the profile corresponding to equal angular movements of the pinion will be equal. In order to derive formulas for the specific sliding of a rack meshing with a pinion, we will start from the pitch line, where we know that pure rolling occurs. There, the length of the infinitely small portion of the profile of the rack that corresponds to the mating portion of the profile of the pinion is equal to the length of the profile of the pinion at that point. The corresponding radius of curvature on the pinion profile at that point is equal to $a_1 \tan \alpha$ (see Fig. 28). Thus, we can write the following formulas for the specific sliding of a rack and pinion:

$$\text{Specific sliding on pinion} = \frac{b_1 - a_1 \tan \alpha}{b_1} \tag{29}$$

$$\text{Specific sliding on rack} = \frac{a_1 \tan \alpha - b_1}{a_1 \tan \alpha} \tag{30}$$

Undercutting of Involute.—As the involute curve stops at the base circle, no involute action can take place below it. If a straight-sided rack with sharp corners acts against the involute, and these corners extend too far below the base circle, interference develops unless the tooth is undercut, as shown in Fig. 29. The looped curve shows the path of the sharp corner of the rack tooth. The sharp corner of the rack will not only undercut the

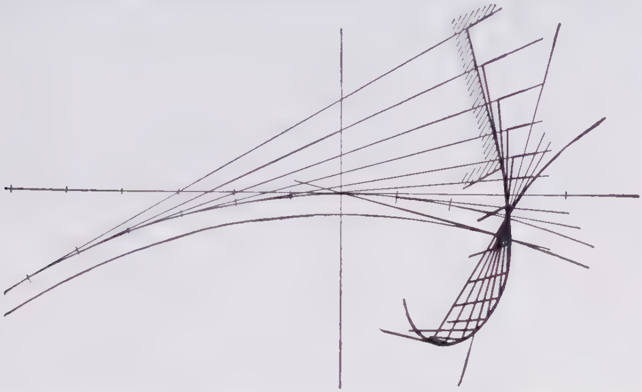

FIG. 29.—Undercutting of involute tooth by rack extending below the base circle.

tooth below the base circle but will also remove the lower part of the involute profile. In order to avoid this interference, the corner of the rack can extend below the base circle only a limited amount. Its bottom edge must not reach below the line where the line of action is tangent to the base circle.

In Fig. 30, let

A = minimum distance between bottom of straight-sided rack tooth with sharp corners and center of base circle
a = radius of base circle
R = radius of pitch circle of involute
α = pressure angle

then

$$A = a \cos \alpha = R \cos^2 \alpha \tag{31}$$

In similar manner, if two involutes are acting against each other, their outside diameters must not reach beyond the point

of tangency of the line of action with the base circle, or a similar interference will develop.

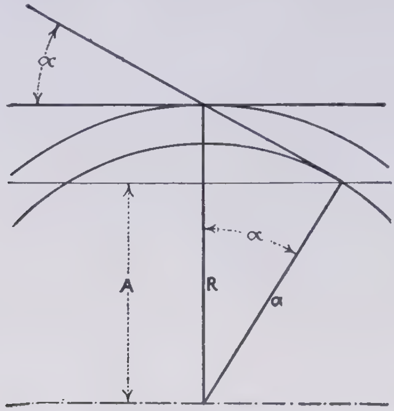

FIG. 30.—Minimum approach of rack to center of base circle.

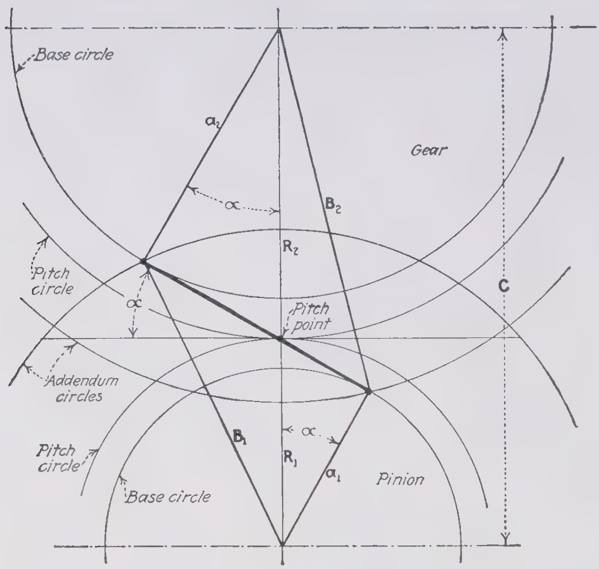

FIG. 31.—Limit of gear outside diameters to avoid interference.

In Fig. 31, let

B_1 = radius of maximum addendum circle of pinion without interference

B_2 = radius of maximum addendum circle of gear without interference

C = center distance

a_1 = radius of base circle of pinion
a_2 = radius of base circle of gear
R_1 = radius of pitch circle of pinion
R_2 = radius of pitch circle of gear
α = pressure angle

Then

$$B_1 = \sqrt{(a_1)^2 + (C \sin \alpha)^2} \tag{32}$$

and

$$B_2 = \sqrt{(a_2)^2 + (C \sin \alpha)^2} \tag{33}$$

TOOTH AND BEARING PRESSURES

The tooth pressure of a gear is the pressure exerted in the direction normal to the tooth profile. This pressure on involute

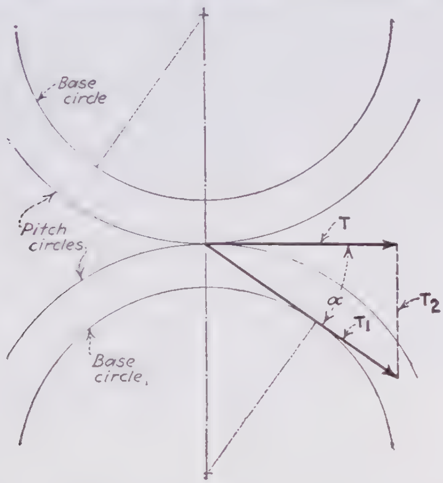

FIG. 32.—Tooth pressure acts along the line of action.

gears is exerted along the line of action. Thus, both the direction and the amount of pressure is constant. This is another feature of the involute form which makes it of great value as a gear-tooth profile.

Referring to Fig. 32, let

T = tangential force acting at pitch circle of involute
T_1 = tooth pressure
α = pressure angle

then

$$T_1 = \frac{T}{\cos \alpha} \tag{34}$$

The pressure on the bearings for the gears will be found in a similar manner. Referring again to Fig. 32, let

T_2 = force tending to separate axes of gears, then
$T_2 = T \tan \alpha$

This force, which tends to separate the axes of the gears, is often confused with the total bearing pressure. As a matter of fact, it is but one component of that force, the other being equal to T. Thus, the total bearing pressure for a single pair of gears becomes

$$\text{Bearing pressure} = \sqrt{T^2 + T_2^2} = T_1 = \frac{T}{\cos \alpha} \qquad (35)$$

The total bearing pressure and the tooth pressure on a single pair of gears are equal. When trains of gears are involved, the bearing pressures will be the resultant of the combinations of tooth pressures, referred to the axis of each gear, and must be determined from the combined components of these forces.

A few computations will show that there is only a small difference in tooth pressure or bearing pressure between pressure angles of 14½ and 25 deg.; these angles cover the range most generally used.

For example, let

$$T = 1{,}000 \text{ lb.}$$
$$\alpha = 14\tfrac{1}{2} \text{ deg.}$$

then

$$T_1 = \frac{1{,}000}{\cos 14\tfrac{1}{2} \text{ deg.}} = 1{,}035 \text{ lb.}$$

Let

$$T = 1{,}000 \text{ lb.}$$

and

$$\alpha = 25 \text{ deg.}$$

then

$$T_1 = \frac{1{,}000}{\cos 25 \text{ deg.}} = 1{,}103 \text{ lb.}$$

The pressures are increased less than 7 per cent when the pressure angles are increased from 14½ to 25 deg. The greatest effect on the bearing pressure, caused by a change in the pressure angle, is a change in its direction.

CHAPTER III

INVOLUTE TRIGONOMETRY

The involute curve has many properties which make it extremely valuable as a gear-tooth form. In practice, full advantage of these properties is not always obtained, because the method of calculating involute tooth sizes is not generally known. These calculations are sometimes complex, but they are not difficult once the few simple fundamentals have been mastered. It is no more difficult to calculate involute tooth sizes than it is to calculate plane triangles. In fact, an involute gear tooth is a form of triangle; the base is an arc of a circle, while the two similar sides are involute curves. We will, therefore, call the process of calculating involute tooth sizes "involute trigonometry."

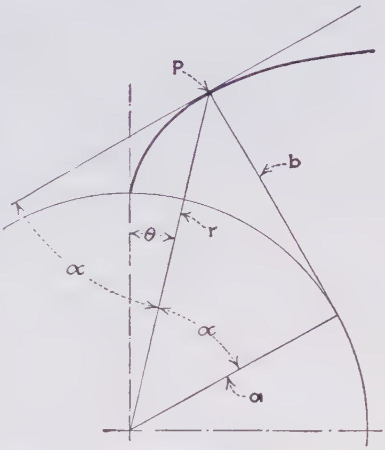

FIG. 33.—Development of the involute curve.

In Fig. 33 is shown the involute curve. We have derived its equation, which follows, in the preceding chapter. Let

a = radius of base circle
r = length of radius vector
θ = vectorial angle
α = angle between radius vector and a radial line to point of tangency of generating line with base circle (this angle on gears is often known as the "pressure angle")

Then

$$\theta = \tan \alpha - \alpha \tag{36}$$

and

$$r = \frac{a}{\cos \alpha} \tag{37}$$

These two equations give the fundamental relationships by means of which involute tooth sizes are readily calculated. It will be seen from Eq. (36) that there is a fixed relationship between the two angles α and θ. The most convenient form of the value of θ is in circular measure, or radians, and its value can be obtained by subtracting the angle α (in radians) from its tangent, as expressed by Eq. (36). The value θ for any angle α will be called the "involute function" of α and will be expressed as "inv α" in the equations that follow. Equation (36) has been tabulated, and these tables of involute functions will be used the same as any other trigometric tables. These tables of involute functions, together with standard trigometric tables, enable all involute calculations to be made in a similar manner to plane triangles.

The further consideration of involute trigonometry will be in the form of a series of problems. First the necessary equations will be derived and then a definite problem will be solved.

PROBLEM 6.—*Given the tooth thickness and angle α (pressure angle) of an involute gear at a given diameter, to determine its tooth thickness at any other diameter.*

Referring to Fig. 34, where

T_1 = the arc thickness of tooth
α_1 = given angle (which would be the *pressure angle* if the given diameter were the *pitch diameter*)
r_1 = radius of given diameter
T_2 = arc thickness of tooth at r_2
r_2 = radius where tooth thickness is required
α_2 = pressure angle at r_2

As the tooth form is symmetrical, we will deal with the half thickness. The half thickness of the tooth at r_1 in circular measure, or radians, equals $T_1/2r_1$.

The half thickness of the tooth in circular measure at the base circle equals $T_1/2r_1 + \text{inv } \alpha_1$. The value of inv α_1 is taken from the table of involute functions.

The half thickness of the tooth in circular measure, or radians, at radius r_2 is equal to its thickness, in radians, on the base circle minus the involute function of α_2. The value of α_2 can be determined from Eq. (37), as follows: Transposing Eq. (37), we have

$$a = r_1 \cos \alpha_1 \tag{38}$$

$$\cos \alpha_2 = \frac{a}{r_2} = \frac{r_1 \cos \alpha_1}{r_2} \tag{39}$$

We have seen before that

$$\frac{T_2}{2r_2} = \frac{T_1}{2r_1} + \text{inv } \alpha_1 - \text{inv } \alpha_2$$

whence,

$$T_2 = 2r_2\left(\frac{T_1}{2r_1} + \text{inv } \alpha_1 - \text{inv } \alpha_2\right) \tag{40}$$

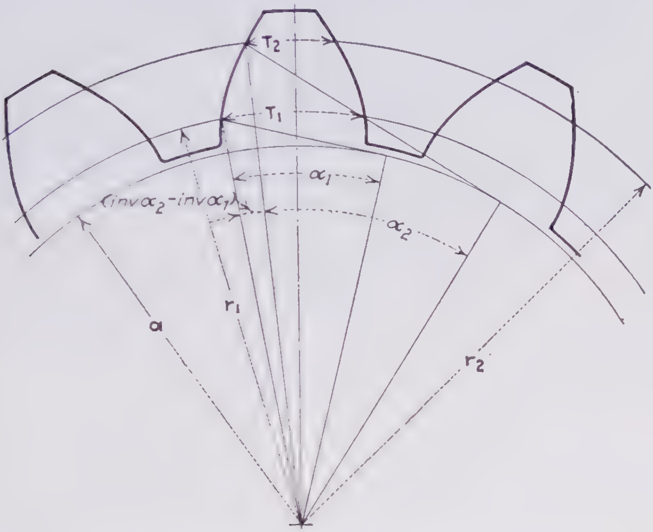

FIG. 34.—Determination of the tooth thickness at various diameters.

Thus, to determine the tooth thickness of an involute gear at any diameter, Eqs. (38), (39), and (40) would be solved in the order given. As an example, let

$$T_1 = 0.2618 \text{ in.}$$
$$\alpha_1 = 20 \text{ deg.}$$
$$r_1 = 2.0000 \text{ in.}$$
$$r_2 = 2.1250 \text{ in.}$$

From Eq. (38), we have

$$a = 2 \cos 20 \text{ deg.} = 1.8794 \text{ in.}$$

From Eq. (39), we have

$$\cos \alpha_2 = \frac{1.8794}{2.1250} = 0.88441$$

whence

$$\alpha_2 = 27° - 49' - 13''$$

From Eq. (40), we have

$$T_2 = 4.250\left[\frac{0.2618}{4.0000} + \text{inv } 20^\circ - \text{inv } (27^\circ - 49' - 13'')\right]$$

From the table of involute functions, we get

$$\text{inv } 20 \text{ deg.} = 0.014904 \text{ radians}$$
$$\text{inv } (27 \text{ deg. } 49 \text{ min. } 13 \text{ sec.}) = 0.042138 \text{ radians}$$

whence

$$T_2 = 4.250(0.06545 + 0.014904 - 0.042138) = 0.1624 \text{ in.}$$

PROBLEM 7.—*Given the tooth thickness at one diameter, to determine the diameter where the tooth becomes pointed.*

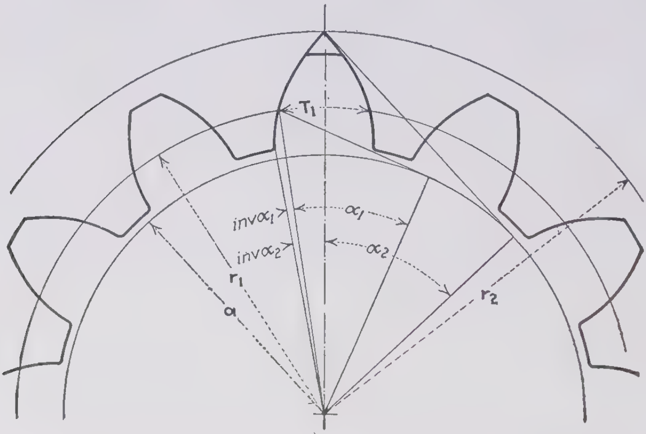

FIG. 35.—Determination of the diameter at which the involute tooth becomes pointed.

Referring to Fig. 35, let

r_1 = given radius
α_1 = pressure angle at radius r_1
T_1 = tooth thickness at r_1
r_2 = radius where T_2 equals zero

Referring to Eq. (40), and making T_2 equal to zero, we should have

$$2r_2\left(\frac{T_1}{2r_1} + \text{inv } \alpha_1 - \text{inv } \alpha_2\right) = 0$$

Dividing by $2r_2$, we have

$$\frac{T_1}{2r_1} + \text{inv } \alpha_1 - \text{inv } \alpha_2 = 0$$

Transposing, this becomes

$$\text{inv } \alpha_2 = \frac{T_1}{2r_1} + \text{inv } \alpha_1 \tag{41}$$

From Eq. (37), we have

$$r_2 = \frac{a}{\cos \alpha_2} = \frac{r_1 \cos \alpha_1}{\cos \alpha_2} \tag{42}$$

Thus, to determine the radius of an involute gear where the tooth would become pointed, Eqs. (41) and (42) would be solved in the order given. As an example, let

$$r_1 = 2.0000 \text{ in.}$$
$$\alpha_1 = 20 \text{ deg.}$$
$$T_1 = 0.2618 \text{ in.}$$

From Eq. (41), we have

$$\text{inv } \alpha_2 = \frac{0.2618}{4.0000} + \text{inv } \alpha_1 = 0.06545 + 0.014904 = 0.08035 \text{ radians}$$

From the table of involute functions, we find that

$$\alpha_2 = 33 \text{ deg. } 54 \text{ min. } 22 \text{ sec.}$$

From Eq. (42), we have

$$r_2 = \frac{2 \cos 20 \text{ deg.}}{\cos 33 \text{ deg. } 54 \text{ min. } 22 \text{ sec.}} = \frac{2 \times 0.93969}{0.82995} = 2.2644 \text{ in.}$$

PROBLEM 8.—*Given the tooth thicknesses of a pair of mating gears, to determine the center distance at which they will mesh tightly.*

Referring to Fig. 36, let

r_1 = radius of smaller gear where tooth thickness is known
t_1 = tooth thickness at r_1
n = number of teeth in smaller gear
α_1 = pressure angle at r_1 and R_1
R_1 = radius of larger gear where tooth thickness is known
T_1 = tooth thickness at R_1
N = number of teeth in larger gear
r_2 = pitch radius of smaller gear when in mesh
t_2 = tooth thickness at r_2
α_2 = pressure angle at r_2 and R_2
R_2 = pitch radius of larger gear when in mesh

TABLE I.—VALUES OF THE INVOLUTE FUNCTIONS OF α. INV α = TAN $\alpha - \alpha = \theta$ RADIANS

Minutes	0°	1°	2°	3°
0	0.00000 00000 00	0.00000 177	0.00001 418	0.00004 790
1	00 08	186	454	871
2	00 66	196	491	952
3	01 22	205	528	0.00005 034
4	05 25	215	565	117
5	10 26	225	603	201
6	17 72	236	642	286
7	28 14	247	682	372
8	42 01	258	722	458
9	59 81	270	762	546
10	82 05	281	804	634
11	0.00000 00109 20	0.00000 294	0.00001 846	0.00005 724
12	141 78	306	888	814
13	180 26	319	931	906
14	225 14	333	975	998
15	276 91	346	0.00002 020	0.00006 091
16	336 06	360	065	186
17	403 10	375	111	281
18	478 50	389	158	377
19	562 76	404	205	474
20	656 38	420	253	573
21	0.00000 00759 84	0.00000 436	0.00002 301	0.00006 672
22	0873 64	452	351	772
23	0998 27	469	401	873
24	1134 23	486	452	975
25	1281 99	504	503	0.00007 078
26	1442 07	522	555	183
27	1614 95	540	608	288
28	1802 12	559	662	394
29	2001 08	579	716	501
30	2215 31	598	771	610
31	0.00000 02444 31	0.00000 618	0.00002 827	0.00007 719
32	2688 57	639	884	829
33	2948 59	660	941	941
34	3224 86	682	999	0.00008 053
35	3517 87	704	0.00003 058	167
36	3828 10	726	117	281
37	4156 07	749	178	397
38	4502 24	772	239	514
39	4867 13	796	301	632
40	5251 22	821	364	751
41	0.00000 05655 01	0.00000 846	0.00003 427	0.00008 871
42	06078 98	871	491	992
43	06523 63	897	556	0.00009 114
44	06989 46	923	622	237
45	07476 95	950	689	362
46	07986 60	978	757	487
47	08518 89	0.00001 005	825	614
48	09074 33	034	894	742
49	09653 41	063	964	870
50	10256 61	092	0.00004 035	0.00010 000
51	0.00000 10884 43	0.00001 123	0.00004 107	0.00010 132
52	11537 37	153	179	264
53	12215 91	184	252	397
54	12920 56	216	327	532
55	13651 79	248	402	668
56	14410 11	281	478	805
57	15196 00	315	554	943
58	16009 97	349	632	0.00011 082
59	16852 50	383	711	223
60	0.00000 17724 08	0.00001 418	0.00004 790	0.00011 364

TABLE I.—VALUES OF THE INVOLUTE FUNCTIONS OF α. INV α = TAN $\alpha - \alpha = \theta$ RADIANS (*Continued*)

Minutes	4°	5°	6°	7°
0	0.00011 364	0.00022 220	0.00038 45	0.00061 15
1	507	443	8 77	1 59
2	651	668	9 09	2 03
3	796	894	9 42	2 48
4	943	0.00023 123	9 75	2 92
5	0.00012 090	352	0.00040 08	3 37
6	239	583	0 41	3 82
7	389	816	0 74	4 27
8	541	0.00024 049	1 08	4 73
9	693	284	1 41	5 18
10	847	522	1 75	5 64
11	0.00013 002	0.00024 761	0.00042 09	0.00066 10
12	158	0.00025 001	2 44	6 57
13	316	243	2 78	7 03
14	474	486	3 13	7 50
15	634	731	3 47	7 97
16	796	977	3 82	8 44
17	958	0.00026 225	4 17	8 92
18	0.00014 122	474	4 53	9 39
19	287	726	4 88	9 87
20	453	978	5 24	0.00070 35
21	0.00014 621	0.00027 233	0.00045 60	0.00070 83
22	790	489	5 96	1 32
23	960	746	6 32	1 81
24	0.00015 132	0.00028 005	6 69	2 30
25	305	266	7 06	2 79
26	479	528	7 43	3 28
27	655	792	7 80	3 78
28	831	0.00029 058	8 17	4 28
29	0.00016 010	325	8 54	4 78
30	189	594	8 92	5 28
31	0.00016 370	0.00029 864	0.00049 30	0.00075 79
32	552	0.00030 137	9 68	6 29
33	736	410	0.00050 06	6 80
34	921	686	0 45	7 32
35	0.00017 107	963	0 83	7 83
36	294	0.00031 242	1 22	8 35
37	483	522	1 61	8 87
38	674	804	2 00	9 39
39	866	0.00032 088	2 40	9 91
40	0.00018 059	374	2 80	0.00080 44
41	0.00018 253	0.00032 661	0.00053 19	0.00080 96
42	449	950	3 59	1 50
43	646	0.00033 241	4 00	2 03
44	845	533	4 40	2 56
45	0.00019 045	827	4 81	3 10
46	247	0.00034 123	5 22	3 64
47	450	421	5 63	4 18
48	654	720	6 04	4 73
49	860	0.00035 021	6 45	5 27
50	0.00020 067	324	6 87	5 82
51	0.00020 276	0.00035 628	0.00057 29	0.00086 38
52	486	934	7 71	6 93
53	698	0.00036 242	8 13	7 49
54	911	552	8 56	8 05
55	0.00021 125	864	8 98	8 61
56	341	0.00037 177	9 41	9 17
57	559	492	9 85	9 74
58	778	809	0.00060 28	0.00090 31
59	998	0.00038 128	0 71	0 88
60	0.00022 220	0.00038 448	0.00061 15	0.00091 45

TABLE I.—VALUES OF THE INVOLUTE FUNCTIONS OF α. INV α = TAN $\alpha - \alpha = \theta$ RADIANS (*Continued*)

Minutes	8°	9°	10°	11°
0	0.00091 45	0.00130 48	0.00179 41	0.00239 41
1	2 03	1 21	0.00180 31	0.00240 51
2	2 60	1 95	1 22	1 61
3	3 18	2 68	2 13	2 72
4	3 77	3 42	3 05	3 83
5	4 35	4 16	3 97	4 95
6	4 94	4 91	4 89	6 07
7	5 53	5 66	5 81	7 19
8	6 12	6 41	6 74	8 31
9	6 72	7 16	7 67	9 44
10	7 32	7 92	8 60	0.00250 57
11	0.00097 92	0.00138 68	0.00189 54	0.00251 71
12	8 52	9 44	0.00190 48	2 85
13	9 13	0.00140 20	1 42	3 99
14	9 73	0 97	2 37	5 13
15	0.00100 34	1 74	3 32	6 28
16	0 96	2 51	4 27	7 44
17	1 57	3 29	5 23	8 59
18	2 19	4 07	6 19	9 75
19	2 81	4 85	7 15	0.00260 91
20	3 43	5 63	8 12	2 08
21	0.00104 06	0.00146 42	0.00199 09	0.00263 25
22	4 69	7 21	0.00200 06	4 43
23	5 32	8 00	1 03	5 60
24	5 95	8 80	2 01	6 78
25	6 59	9 60	2 99	7 97
26	7 22	0.00150 40	3 98	9 16
27	7 86	1 20	4 97	0.00270 35
28	8 51	2 01	5 96	1 54
29	9 15	2 82	6 95	2 74
30	9 80	3 63	7 95	3 94
31	0.00110 45	0.00154 45	0.00208 95	0.00275 15
32	1 11	5 27	9 95	6 36
33	1 76	6 09	0.00210 96	7 57
34	2 42	6 91	1 97	8 79
35	3 08	7 74	2 99	0.00280 01
36	3 75	8 57	4 00	1 23
37	4 41	9 41	5 02	2 46
38	5 08	0.00160 24	6 05	3 69
39	5 75	1 08	7 07	4 93
40	6 43	1 93	8 10	6 16
41	0.00117 11	0.00162 77	0.00219 14	0.00287 41
42	7 79	3 62	0.00220 17	8 65
43	8 47	4 47	1 21	9 90
44	9 15	5 33	2 26	0.00291 15
45	9 84	6 18	3 30	2 41
46	0.00120 53	7 04	4 35	3 67
47	1 22	7 91	5 41	4 94
48	1 92	8 77	6 47	6 20
49	2 62	9 64	7 53	7 47
50	3 32	0.00170 51	8 59	8 75
51	0.00124 02	0.00171 39	0.00229 66	0.00300 03
52	4 73	2 27	0.00230 73	1 31
53	5 44	3 15	1 80	2 60
54	6 15	4 03	2 88	3 89
55	6 87	4 92	3 96	5 18
56	7 58	5 81	5 04	6 48
57	8 30	6 71	6 13	7 78
58	9 03	7 60	7 22	9 08
59	9 75	8 50	8 31	0.00310 39
60	0.00130 48	0.00179 41	0.00239 41	0.00311 71

TABLE I.—VALUES OF THE INVOLUTE FUNCTIONS OF α. INV α = TAN $\alpha - \alpha = \theta$ RADIANS (*Continued*)

Minutes	12°	13°	14°	15°
0	0.00311 71	0.00397 54	0.00498 19	0.00614 98
1	3 02	9 09	0.00500 00	7 07
2	4 34	0.00400 65	1 82	9 17
3	5 67	2 21	3 64	0.00621 27
4	6 99	3 77	5 46	3 37
5	8 32	5 34	7 29	5 48
6	9 66	6 92	9 12	7 60
7	0.00321 00	8 49	0.00510 96	9 72
8	2 34	0.00410 08	2 80	0.00631 84
9	3 69	1 66	4 65	3 97
10	5 04	3 25	6 50	6 11
11	0.00326 39	0.00414 85	0.00518 35	0.00638 25
12	7 75	6 44	0.00520 22	0.00640 39
13	9 11	8 05	2 08	2 54
14	0.00330 48	9 65	3 95	4 70
15	1 85	0.00421 26	5 82	6 86
16	3 22	2 88	7 70	9 02
17	4 60	4 50	9 58	0.00651 19
18	5 98	6 12	0.00531 47	3 37
19	7 36	7 75	3 36	5 55
20	8 75	9 38	5 26	7 73
21	0.00340 14	0.00431 02	0.00537 16	0.00659 92
22	1 54	2 66	9 07	0.00662 11
23	2 94	4 30	0.00540 98	4 31
24	4 34	5 95	2 90	6 52
25	5 75	7 60	4 82	8 73
26	7 16	9 26	6 74	0.00670 94
27	8 58	0.00440 92	8 67	3 16
28	0.00350 00	2 59	0.00550 60	5 39
29	1 42	4 26	2 54	7 62
30	2 85	5 93	4 48	9 85
31	0.00354 28	0.00447 61	0.00556 43	0.00682 09
32	5 72	9 29	8 38	4 34
33	7 16	0.00450 98	0.00560 34	6 59
34	8 60	2 67	2 30	8 84
35	0.00360 05	4 37	4 27	0.00691 10
36	1 50	6 07	6 24	3 37
37	2 96	7 77	8 22	5 64
38	4 41	9 48	0.00570 20	7 91
39	5 88	0.00461 20	2 18	0.00700 19
40	7 35	2 91	4 17	2 48
41	0.00368 82	0.00464 64	0.00576 17	0.00704 77
42	0.00370 29	6 36	8 17	7 06
43	1 77	8 09	0.00580 17	9 36
44	3 26	9 83	2 18	0.00711 67
45	4 74	0.00471 57	4 20	3 98
46	6 23	3 31	6 22	2 30
47	7 73	5 06	8 24	8 62
48	9 23	6 81	0.00590 28	0.00720 95
49	0.00380 73	8 57	2 30	3 28
50	2 24	0.00480 33	4 34	5 61
51	0.00383 75	0.00482 10	0.00596 38	0.00727 96
52	5 27	3 87	8 43	0.00730 30
53	6 79	5 64	0.00600 48	2 66
54	8 31	7 42	2 54	5 01
55	9 84	9 21	4 60	7 38
56	0.00391 37	0.00490 99	6 67	9 75
57	2 91	2 79	8 74	0.00742 12
58	4 45	4 58	0.00610 81	4 50
59	5 99	6 39	2 89	6 88
60	0.00397 54	0.00498 19	0.00614 98	0.00749 27

TABLE I.—VALUES OF THE INVOLUTE FUNCTIONS OF α. INV α = TAN $\alpha - \alpha = \theta$ RADIANS (*Continued*)

Minutes	16°	17°	18°	19°	20°
0	0.00749 3	0.00902 5	0.01076 0	0.01271 5	0.01490 4
1	51 7	05 2	79 1	75 0	94 3
2	54 1	07 9	82 2	78 4	98 2
3	56 5	10 7	85 3	81 9	0.01502 0
4	58 9	13 4	88 4	85 4	05 9
5	61 3	16 1	91 5	88 8	09 8
6	63 7	18 9	94 6	92 3	13 7
7	66 1	21 6	97 7	95 8	17 6
8	68 6	24 4	0.01100 8	99 3	21 5
9	71 0	27 2	03 9	0.01302 8	25 4
10	73 5	29 9	07 1	06 3	29 3
11	0.00775 9	0.00932 7	0.01110 2	0.01309 8	0.01533 3
12	78 4	35 5	13 3	13 4	37 2
13	80 8	38 3	16 5	16 9	41 1
14	83 3	41 1	19 6	20 4	45 1
15	85 7	43 9	22 8	24 0	49 0
16	88 2	46 7	26 0	27 5	53 0
17	90 7	49 5	29 1	31 1	57 0
18	93 2	52 3	32 3	34 6	60 9
19	95 7	55 2	35 5	38 2	64 9
20	98 2	58 0	38 7	41 8	68 9
21	0.00800 7	0.00960 8	0.01141 9	0.01345 4	0.01572 9
22	03 2	63 7	45 1	49 0	76 9
23	5 7	66 5	48 3	52 6	80 9
24	08 2	69 4	51 5	56 2	85 0
25	10 7	72 2	54 7	59 8	89 0
26	13 3	75 1	58 0	63 4	93 0
27	15 8	78 0	61 2	67 0	97 1
28	18 3	80 8	64 4	70 7	0.01601 1
29	20 9	83 7	67 7	74 3	05 2
30	23 4	86 6	70 9	77 9	09 2
31	0.00826 0	0.00989 5	0.01174 2	0.01381 6	0.01613 3
32	28 5	92 4	77 5	85 2	17 4
33	31 1	95 3	80 7	88 9	21 5
34	33 7	98 2	84 0	92 6	25 5
35	36 2	0.01001 2	87 3	96 3	29 6
36	38 8	04 1	90 6	99 9	33 7
37	41 4	07 0	93 9	0.01403 6	37 9
38	44 0	09 9	97 2	07 3	42 0
39	46 6	12 9	0.01200 5	11 0	46 1
40	49 2	15 8	03 8	14 8	50 2
41	0.00851 8	0.01018 8	0.01207 1	0.01418 5	0.01654 4
42	54 4	21 7	10 5	22 2	58 5
43	57 1	24 7	13 8	25 9	62 7
44	59 7	27 7	17 2	29 7	66 9
45	62 3	30 7	20 5	33 4	71 0
46	65 0	33 6	23 9	37 2	75 2
47	67 6	36 6	27 2	40 9	79 4
48	70 2	39 6	30 6	44 7	83 6
49	72 9	42 6	34 0	48 5	87 8
50	75 6	45 6	37 3	52 3	92 0
51	0.00878 2	0.01048 6	0.01240 7	0.01456 0	0.01696 2
52	80 9	51 7	44 1	59 8	0.01700 4
53	83 6	54 7	47 5	63 6	04 7
54	86 3	57 7	50 9	67 4	08 9
55	88 9	60 8	54 3	71 3	13 2
56	91 6	63 8	57 8	75 1	17 4
57	94 3	66 9	61 2	78 9	21 7
58	97 0	69 9	64 6	82 7	25 9
59	99 8	73 0	68 1	86 6	30 2
60	0.00902 5	0.01076 0	0.01271 5	0.01490 4	0.01734 5

TABLE I.—VALUES OF THE INVOLUTE FUNCTIONS OF α. INV α = TAN $\alpha - \alpha = \theta$ RADIANS (*Continued*)

Minutes	21°	22°	23°	24°	25°
0	0.01734 5	0.02005 4	0.02304 9	0.02635 0	0.02997 5
1	38 8	10 1	10 2	40 7	0.03003 9
2	43 1	14 9	15 4	46 5	10 2
3	47 4	19 7	20 7	52 3	16 6
4	51 7	24 4	25 9	58 1	22 9
5	56 0	29 2	31 2	63 9	29 3
6	60 3	34 0	36 5	69 7	35 7
7	64 7	38 8	41 8	75 6	42 0
8	69 0	43 6	47 1	81 4	48 4
9	73 4	48 4	52 4	87 2	54 9
10	77 7	53 3	57 7	93 1	61 3
11	0.01782 1	0.02058 1	0.02363 1	0.02698 9	0.03067 7
12	86 5	62 9	68 4	0.02704 8	74 1
13	90 8	67 8	73 8	10 7	80 6
14	95 2	72 6	79 1	16 6	87 0
15	99 6	77 5	84 5	22 5	93 5
16	0.01804 0	82 4	89 9	28 4	0.03100 0
17	08 4	87 3	95 2	34 3	06 5
18	12 9	92 1	0.02400 6	40 2	13 0
19	17 3	97 0	06 0	46 2	19 5
20	21 7	0.02101 9	11 4	52 1	26 0
21	0.01826 2	0.02106 9	0.02416 9	0.02758 1	0.03132 5
22	30 6	11 8	22 3	64 0	39 0
23	35 1	16 7	27 7	70 0	45 6
24	39 5	21 7	33 2	76 0	52 1
25	44 0	26 6	38 6	82 0	58 7
26	48 5	31 6	44 1	88 0	65 3
27	53 0	36 5	49 5	94 0	71 8
28	57 5	41 5	55 0	0.02800 0	78 4
29	62 0	46 5	60 5	06 0	85 0
30	66 5	51 4	66 0	12 1	91 7
31	0.01871 0	0.02156 4	0.02471 5	0.02818 1	0.03198 3
32	75 5	61 4	77 0	24 2	0.03204 9
33	80 0	66 5	82 5	30 2	11 6
34	84 6	71 5	88 1	36 3	18 2
35	89 1	76 5	93 6	42 4	24 9
36	93 7	81 5	99 2	48 5	31 5
37	98 3	86 6	0.02504 7	54 6	38 2
38	0.01902 8	91 6	10 3	60 7	44 9
39	07 4	96 7	15 9	66 8	51 6
40	12 0	0.02201 8	21 4	72 9	58 3
41	0.01916 6	0.02206 8	0.02527 0	0.02879 1	0.03265 1
42	21 2	11 9	32 6	85 2	71 8
43	25 8	17 0	38 2	91 4	78 5
44	30 4	22 1	43 9	97 6	85 3
45	35 0	27 2	49 5	0.02903 7	92 0
46	39 7	32 4	55 1	09 9	98 8
47	44 3	37 5	60 8	16 1	0.03305 6
48	49 0	42 6	66 4	22 3	12 4
49	53 6	47 8	72 1	28 5	19 2
50	58 3	52 9	77 8	34 8	26 0
51	0.01963 0	0.02258 1	0.02583 4	0.02941 0	0.03332 8
52	67 6	63 3	89 1	47 2	39 7
53	72 3	68 4	94 8	53 5	46 5
54	77 0	73 6	0.02600 5	59 8	53 4
55	81 7	78 8	06 2	66 0	60 2
56	86 4	84 0	12 0	72 3	67 1
57	91 2	89 2	17 7	78 6	74 0
58	95 9	94 4	23 5	84 9	80 9
59	0.02000 7	99 7	29 2	91 2	87 8
60	0.02005 4	0.02304 9	0.02635 0	0.02997 5	0.03394 7

TABLE I.—VALUES OF THE INVOLUTE FUNCTIONS OF α. INV α = TAN $\alpha - \alpha = \theta$ RADIANS (*Continued*)

Minutes	26°	27°	28°	29°	30°
0	0.03394 7	0.03828 7	0.04301 7	0.04816 4	0.05375 1
1	0.03401 6	36 2	10 0	25 3	84 9
2	08 6	43 8	18 2	34 3	94 6
3	15 5	51 4	26 4	43 2	0.05404 3
4	22 5	59 0	34 7	52 2	14 0
5	29 4	66 6	43 0	61 2	23 8
6	36 4	74 2	51 3	70 2	33 6
7	43 4	81 8	59 6	79 2	43 3
8	50 4	89 4	67 9	88 3	53 1
9	57 4	97 1	76 2	97 3	62 9
10	64 4	0.03904 7	84 5	0.04906 4	72 8
11	0.03471 4	0.03912 4	0.04392 9	0.04915 4	0.05482 6
12	78 5	20 1	0.04401 2	24 5	92 4
13	85 5	27 8	09 6	33 6	0.05502 3
14	92 6	35 5	18 0	42 7	12 2
15	99 7	43 2	26 4	51 8	22 1
16	0.03506 7	50 9	34 8	60 9	32 0
17	13 8	58 6	43 2	70 1	41 9
18	20 9	66 4	51 6	79 2	51 8
19	28 0	74 1	60 1	88 4	61 7
20	35 2	81 9	68 5	97 6	71 7
21	0.03542 3	0.03989 7	0.04477 0	0.05006 8	0.05581 7
22	49 4	97 4	85 5	16 0	91 6
23	56 6	0.04005 2	93 9	25 2	0.05601 6
24	63 7	13 1	0.04502 4	34 4	11 6
25	70 9	20 9	11 0	43 7	21 7
26	78 1	28 7	19 5	52 9	31 7
27	85 3	36 6	28 0	62 2	41 7
28	92 5	44 4	36 6	71 5	51 8
29	99 7	52 3	45 1	80 8	61 9
30	0.03606 9	60 2	53 7	90 1	72 0
31	0.03614 2	0.04068 0	0.04562 3	0.05099 4	0.05682 1
32	21 4	75 9	70 9	0.05108 7	92 2
33	28 7	83 9	79 5	18 1	0.05702 3
34	35 9	91 8	88 1	27 4	12 4
35	43 2	99 7	96 7	36 8	22 6
36	50 5	0.04107 6	0.04605 4	46 2	32 8
37	57 8	15 6	14 0	55 6	42 9
38	65 1	23 6	22 7	65 0	53 1
39	72 4	31 6	31 3	74 4	63 3
40	79 8	39 5	40 0	83 8	73 6
41	0.03687 1	0.04147 5	0.04648 7	0.05193 3	0.05783 8
42	94 5	55 6	57 5	0.05202 7	94 0
43	0.03701 8	63 6	66 2	12 2	0.05804 3
44	09 2	71 6	74 9	21 7	14 6
45	16 6	79 7	83 7	31 2	24 9
46	24 0	87 7	92 4	40 7	35 2
47	31 4	95 8	0.04701 2	50 2	45 5
48	38 8	0.04203 9	10 0	59 7	55 8
49	46 2	12 0	18 8	69 3	66 2
50	53 7	20 1	27 6	78 8	76 5
51	0.03761 1	0.04228 2	0.04736 4	0.05288 4	0.05886 9
52	68 6	36 3	45 2	98 0	97 3
53	76 1	44 4	54 1	0.05307 6	0.05907 7
54	83 5	52 6	63 0	17 2	18 1
55	91 0	60 7	71 8	26 8	28 5
56	98 5	68 9	80 7	36 5	39 0
57	0.03806 0	77 1	89 6	46 1	49 4
58	13 6	85 3	98 5	55 8	59 9
59	21 1	93 5	0.04807 4	65 5	70 4
60	0.03828 7	0.04301 7	0.04816 4	0.05375 1	0.05980 9

TABLE I.—VALUES OF THE INVOLUTE FUNCTIONS OF α. INV α = TAN $\alpha - \alpha = \theta$ RADIANS (*Continued*)

Minutes	31°	32°	33°	34°	35°
0	0.05980 9	0.06636 4	0.07344 9	0.08109 7	0.08934 2
1	91 4	47 8	57 2	22 9	48 5
2	0.06001 9	59 1	69 5	36 2	62 8
3	12 4	70 5	81 8	49 4	77 1
4	23 0	81 9	94 1	62 7	91 4
5	33 5	93 4	0.07406 4	76 0	0.09005 8
6	44 1	0.06704 8	18 8	89 4	20 1
7	54 7	16 3	31 2	0.08202 7	34 5
8	65 3	27 7	43 5	16 1	48 9
9	75 9	39 2	55 9	29 4	63 3
10	86 6	50 7	68 4	42 8	77 7
11	0.06097 2	0.06762 2	0.07480 8	0.08256 2	0.09092 2
12	0.06107 9	73 8	93 2	69 7	0.09106 7
13	18 6	85 3	0.07505 7	83 1	21 1
14	29 2	96 9	18 2	96 6	35 6
15	40 0	0.06808 4	30 7	0.08310 0	50 2
16	50 7	20 0	43 2	23 5	64 7
17	61 4	31 6	55 7	37 1	79 3
18	72 1	43 2	68 3	50 6	93 8
19	82 9	54 9	80 8	64 1	0.09208 4
20	93 7	66 5	93 4	77 7	23 0
21	0.06204 5	0.06878 2	0.07606 0	0.08391 3	0.09237 7
22	15 3	89 9	18 6	0.08404 9	52 3
23	26 1	0.06901 6	31 2	18 5	67 0
24	36 9	13 3	43 9	32 1	81 6
25	47 8	25 0	56 5	45 7	96 3
26	58 6	36 7	69 2	59 4	0.09311 1
27	69 5	48 5	81 9	73 1	25 8
28	80 4	60 2	94 6	86 8	40 6
29	91 3	72 0	0.07707 3	0.08500 5	55 3
30	0.06302 2	83 8	20 0	14 2	70 1
31	0.06313 1	0.06995 1	0.07732 8	0.08528 0	0.09384 9
32	24 1	0.07007 5	45 5	41 8	99 8
33	35 0	19 3	58 3	55 5	0.09414 6
34	46 0	31 2	71 1	69 3	29 5
35	57 0	43 0	83 9	83 2	44 3
36	68 0	54 9	96 8	97 0	59 2
37	79 0	66 8	0.07809 6	0.08610 8	74 2
38	90 1	78 7	22 5	24 7	89 1
39	0.06401 1	90 7	35 4	38 6	0.09504 1
40	12 2	0.07102 6	48 3	52 5	19 0
41	0.06423 2	0.07114 6	0.07861 2	0.08666 4	0.09534 0
42	34 3	26 6	74 1	80 4	49 0
43	45 4	38 6	87 1	94 3	64 1
44	56 5	50 6	0.07900 0	0.08708 3	79 1
45	67 7	62 6	13 0	22 3	94 2
46	78 8	74 7	26 0	36 3	0.09609 3
47	90 0	86 7	39 0	50 3	24 4
48	0.06501 2	98 8	52 0	64 4	39 5
49	12 3	0.07210 9	65 1	78 4	54 6
50	23 6	23 0	78 1	92 5	69 8
51	0.06534 8	0.07235 1	0.07991 2	0.08806 6	0.09685 0
52	46 0	47 3	0.08004 3	20 7	0.09700 2
53	57 3	59 4	17 4	34 8	15 4
54	68 5	71 6	30 6	49 0	30 6
55	79 8	83 8	43 7	63 1	45 9
56	91 1	95 9	56 9	77 3	61 1
57	0.06602 4	0.07308 2	70 0	91 5	76 4
58	13 7	20 4	83 2	0.08905 7	91 7
59	25 0	32 6	96 4	20 0	0.09807 1
60	0.06636 4	0.07344 9	0.08109 7	0.08934 2	0.09822 4

TABLE I.—VALUES OF THE INVOLUTE FUNCTIONS OF α. INV α = TAN $\alpha - \alpha = \theta$ RADIANS (*Continued*)

Minutes	36°	37°	38°	39°	40°
0	0.09822	0.10778	0.11806	0.12911	0.14097
1	838	795	824	930	117
2	853	811	842	949	138
3	869	828	859	968	158
4	884	844	877	987	179
5	899	861	895	0.13006	200
6	915	878	913	025	220
7	930	894	931	045	241
8	946	911	949	064	261
9	961	928	957	083	282
10	977	944	985	102	303
11	0.09992	0.10961	0.12003	0.13122	0.14324
12	0.10008	978	021	141	344
13	024	995	039	160	365
14	039	0.11011	057	180	386
15	055	028	075	199	407
16	070	045	093	219	428
17	086	062	111	238	448
18	102	079	129	258	469
19	118	096	147	277	490
20	133	113	165	297	511
21	0.10149	0.11130	0.12184	0.13316	0.14532
22	165	146	202	336	553
23	181	163	220	355	574
24	196	180	238	375	595
25	212	197	257	395	616
26	228	215	275	414	638
27	244	232	293	434	659
28	260	249	312	454	680
29	276	266	330	473	701
30	292	283	348	493	722
31	0.10308	0.11300	0.12367	0.13513	0.14743
32	323	317	385	533	765
33	339	334	404	553	786
34	355	352	422	572	807
35	371	369	441	592	829
36	388	386	459	612	850
37	404	403	478	632	871
38	420	421	496	652	893
39	436	438	515	672	914
40	452	455	534	692	936
41	0.10468	0.11473	0.12552	0.13712	0.14957
42	484	490	571	732	979
43	500	507	590	752	0.15000
44	516	525	608	772	022
45	533	542	627	792	043
46	549	560	646	812	065
47	565	577	664	833	087
48	581	595	683	853	108
49	598	612	702	873	130
50	614	630	721	893	152
51	0.10630	0.11647	0.12740	0.13913	0.15173
52	647	665	759	934	195
53	663	682	778	954	217
54	679	700	797	974	239
55	696	718	815	995	261
56	712	735	834	0.14015	282
57	729	753	853	035	304
58	745	771	872	056	326
59	762	788	891	076	348
60	0.10778	0.11806	0.12911	0.14097	0.15370

TABLE I.—VALUES OF THE INVOLUTE FUNCTIONS OF α. INV α = TAN $\alpha - \alpha = \theta$ RADIANS (*Continued*)

Minutes	41°	42°	43°	44°	45°
0	0.15370	0.16737	0.18202	0.19774	0.21460
1	392	760	228	802	489
2	414	784	253	829	518
3	436	807	278	856	548
4	458	831	304	883	577
5	480	855	329	910	606
6	503	879	355	938	635
7	525	902	380	965	665
8	547	926	406	992	694
9	569	950	431	0.20020	723
10	591	974	457	047	753
11	0.15614	0.16998	0.18482	0.20075	0.21782
12	636	0.17022	508	102	812
13	658	045	534	130	841
14	680	069	559	157	871
15	703	093	585	185	900
16	725	117	611	212	930
17	748	142	637	240	960
18	770	166	662	268	989
19	793	190	688	296	0.22019
20	815	214	714	323	049
21	0.15838	0.17238	0.18740	0.20351	0.22079
22	860	262	766	379	108
23	883	286	792	407	138
24	905	311	818	435	168
25	928	335	844	463	198
26	950	359	870	490	228
27	973	383	896	518	258
28	996	408	922	546	288
29	0.16019	432	948	575	318
30	041	457	975	603	348
31	0.16064	0.17481	0.19001	0.20631	0.22378
32	087	506	027	659	409
33	110	530	053	687	439
34	133	555	080	715	469
35	156	579	106	743	499
36	178	604	132	772	530
37	201	628	159	800	560
38	224	653	185	828	590
39	247	678	212	857	621
40	270	702	238	885	651
41	0.16293	0.17727	0.19265	0.20914	0.22682
42	317	752	291	942	712
43	340	777	318	971	743
44	363	801	344	999	773
45	386	826	371	0.21028	804
46	409	851	398	056	835
47	432	876	424	085	865
48	456	901	451	114	896
49	479	926	478	142	927
50	502	951	505	171	958
51	0.16525	0.17976	0.19532	0.21200	0.22989
52	549	0.18001	558	229	0.23020
53	572	026	585	257	050
54	596	051	612	286	081
55	619	076	639	315	112
56	642	101	666	344	143
57	666	127	693	373	174
58	689	152	720	402	206
59	713	177	747	431	237
60	0.16737	0.18202	0.19774	0.21460	0.23268

TABLE I.—VALUES OF THE INVOLUTE FUNCTIONS OF α. INV α = TAN $\alpha - \alpha = \theta$ RADIANS (*Continued*)

Minutes	46°	47°	48°	49°	50°
0	0.23268	0.25206	0.27285	0.29516	0.31909
1	299	240	321	554	950
2	330	273	357	593	992
3	362	307	393	631	0.32033
4	393	341	429	670	075
5	424	374	465	709	116
6	456	408	501	747	158
7	487	442	538	786	199
8	519	475	574	825	241
9	550	509	610	864	283
10	582	543	646	903	324
11	0.23613	0.25577	0.27683	0.29942	0.32366
12	645	611	719	981	408
13	676	645	755	0.30020	450
14	708	679	792	059	492
15	740	713	828	098	534
16	772	747	865	137	576
17	803	781	902	177	618
18	835	815	938	216	661
19	867	849	975	255	703
20	899	883	0.28012	0.30295	745
21	0.23931	0.25918	0.28048	0.30334	0.32787
22	963	952	085	374	830
23	995	986	122	413	872
24	0.24027	0.26021	159	453	915
25	059	055	196	492	957
26	091	089	233	532	0.33000
27	123	124	270	572	042
28	156	159	307	611	085
29	188	193	344	651	128
30	220	228	381	691	171
31	0.24253	0.26262	0.28418	0.30731	0.33213
32	285	297	455	771	256
33	317	332	493	811	299
34	350	368	530	851	342
35	382	401	567	891	385
36	415	436	605	931	428
37	447	471	642	971	471
38	480	506	680	0.31012	515
39	512	541	717	052	558
40	545	576	755	092	601
41	0.24578	0.26611	0.28792	0.31133	0.33645
42	611	646	830	173	688
43	643	682	868	214	731
44	676	717	906	254	775
45	709	752	943	295	818
46	742	787	981	335	862
47	775	823	0.29019	376	906
48	808	858	057	417	949
49	841	893	095	457	993
50	874	929	133	498	0.34037
51	0.24907	0.26964	0.29171	0.31539	0.34081
52	940	0.27000	209	580	125
53	973	035	247	621	169
54	0.25006	071	286	662	213
55	040	107	324	703	257
56	073	142	362	744	301
57	106	178	400	785	345
58	140	214	439	826	389
59	173	250	477	868	434
60	0.25206	0.27285	0.29516	0.31909	0.34478

TABLE I.—VALUES OF THE INVOLUTE FUNCTIONS OF α. INV α = TAN $\alpha - \alpha = \theta$ RADIANS (*Continued*)

Minutes	51°	52°	53°	54°	55°
0	0.34478	0.37237	0.40202	0.43390	0.46822
1	522	285	253	446	881
2	567	332	305	501	940
3	611	380	356	556	0.47000
4	656	428	407	611	060
5	700	476	459	667	119
6	745	524	511	722	179
7	790	572	562	778	239
8	834	620	614	833	299
9	879	668	666	889	359
10	924	716	717	945	419
11	0.34969	0.37765	0.40769	0.44001	0.47479
12	0.35014	813	821	057	539
13	059	861	873	113	599
14	104	910	925	169	660
15	149	958	977	225	720
16	194	0.38007	0.41030	281	780
17	240	055	082	337	841
18	285	104	134	393	902
19	330	153	187	450	962
20	376	202	239	506	0.48023
21	0.35421	0.38251	0.41292	0.44563	0.48084
22	467	299	344	619	145
23	512	348	397	676	206
24	558	397	450	733	267
25	604	446	502	789	328
26	649	496	555	846	389
27	695	545	608	903	451
28	741	594	661	960	512
29	787	643	714	0.45017	574
30	833	693	767	047	635
31	0.35879	0.38742	0.41820	0.45132	0.48697
32	925	792	874	189	758
33	971	841	927	246	820
34	0.36017	891	980	304	882
35	063	941	0.42034	361	944
36	110	990	087	419	0.49006
37	156	0.39040	141	476	068
38	202	090	194	534	130
39	249	140	248	592	192
40	295	190	302	650	255
41	0.36342	0.39240	0.42355	0.45708	0.49317
42	388	290	409	766	380
43	435	340	463	824	442
44	482	390	517	882	505
45	529	441	571	940	568
46	575	491	625	998	630
47	622	541	680	0.46057	693
48	669	592	734	115	756
49	716	642	788	173	819
50	763	693	843	232	882
51	0.36810	0.39743	0.42897	0.46291	0.49945
52	858	794	952	349	0.50009
53	905	845	0.43006	408	072
54	952	896	061	467	135
55	999	947	116	526	199
56	0.37047	998	171	585	263
57	094	0.40049	225	644	326
58	142	100	280	703	390
59	189	151	335	762	454
60	0.37237	0.40202	0.43390	0.46822	0.50518

TABLE I.—VALUES OF THE INVOLUTE FUNCTIONS OF α. INV α = TAN $\alpha - \alpha = \theta$ RADIANS (*Continued*)

Minutes	56°	57°	58°	59°
0	0.50518	0.54503	0.58804	0.63454
1	582	572	879	534
2	646	641	954	615
3	710	710	0.59028	696
4	774	779	103	777
5	838	849	178	858
6	903	918	253	939
7	967	988	328	0.64020
8	0.51032	0.55057	403	102
9	096	127	479	183
10	161	197	554	265
11	0.51226	0.55267	0.59630	0.64346
12	291	337	705	428
13	356	407	781	510
14	421	477	857	592
15	486	547	933	674
16	551	618	0.60009	756
17	616	688	085	839
18	682	759	161	921
19	747	829	237	0.65004
20	813	900	314	086
21	0.51878	0.55971	0.60390	0.65169
22	944	0.56042	467	252
23	0.52010	113	544	335
24	076	184	620	418
25	141	255	697	501
26	207	326	774	585
27	274	398	851	668
28	340	469	929	752
29	406	541	0.61006	835
30	472	612	083	919
31	0.52539	0.56684	0.61161	0.66003
32	605	756	239	087
33	672	828	316	171
34	739	900	394	255
35	805	972	472	340
36	872	0.57044	550	424
37	939	116	628	509
38	0.53006	188	706	593
39	073	261	785	678
40	141	333	863	763
41	0.53208	0.57406	0.61942	0.66848
42	275	479	0.62020	933
43	343	552	099	0.67019
44	410	625	178	104
45	478	698	257	189
46	546	771	336	275
47	613	844	415	361
48	681	917	494	447
49	749	991	574	532
50	817	0.58064	653	618
51	0.53885	0.58138	0.62733	0.67705
52	954	211	812	791
53	0.54022	285	892	877
54	090	359	972	964
55	159	433	0.63052	0.68050
56	228	507	132	137
57	296	581	212	224
58	365	656	293	311
59	434	730	373	398
60	0.54503	0.58804	0.63454	0.68485

T_2 = tooth thickness at R_2
C_1 = center distance for pressure angle of α_1
C_2 = center distance for pressure angle of α_2

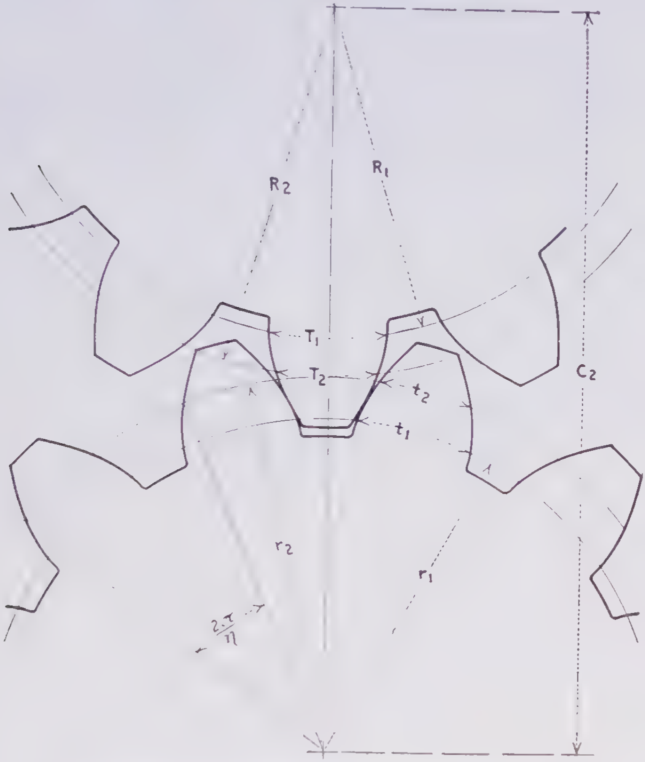

FIG. 36.—Determination of the center distance for tight-meshing involute gears.

From Eq. (40), we get the tooth thicknesses at r_2 and R_2 as follows:

$$t_2 = 2r_2\left(\frac{t_1}{2r_1} + \operatorname{inv}\alpha_1 - \operatorname{inv}\alpha_2\right)$$

$$T_2 = 2R_2\left(\frac{T_1}{2R_1} + \operatorname{inv}\alpha_1 - \operatorname{inv}\alpha_2\right)$$

As the sum of these two tooth thicknesses must be equal to the circular pitch, which, in turn, is equal to the quotient of the circumference of either pitch circle divided by its number of teeth, we have

$$t_2 + T_2 = \frac{2\pi r_2}{n}$$

Furthermore, we know that the pitch diameters of two mating gears are directly proportional to their numbers of teeth, whence we have

$$R_1 = \frac{N}{n}r_1 \text{ and } R_2 = \frac{N}{n}r_2$$

Substituting these values in the equation for T_2, we have

$$T_2 = \frac{2N}{n}r_2\left(\frac{T_1 n}{2Nr_1} + \text{inv}\,\alpha_1 - \text{inv}\,\alpha_2\right)$$

whence,

$$t_2 + T_2 = \frac{2\pi r_2}{n} = 2r_2\left[\frac{t_1}{r_1} + \text{inv}\,\alpha_1 - \text{inv}\,\alpha_2 + \frac{N}{n}\left(\frac{T_1 n}{2Nr_1} + \text{inv}\,\alpha_1 - \text{inv}\,\alpha_2\right)\right]$$

By combining terms, simplifying, and solving for α_2, this equation becomes

$$\text{inv}\,\alpha_2 = \frac{n(t_1 + T_1) - 2\pi r_1}{2r_1(n + N)} + \text{inv}\,\alpha_1 \tag{43}$$

From Eq. (37), we have

$$r_2 = \frac{r_1 \cos \alpha_1}{\cos \alpha_2} \text{ and } R_2 = \frac{R_1 \cos \alpha_1}{\cos \alpha_2}$$

Also, we know that

$$r_1 + R_1 = C_1 \text{ and } r_2 + R_2 = C_2$$

whence,

$$C_2 = \frac{C_1 \cos \alpha_1}{\cos \alpha_2} \tag{44}$$

Thus, to determine the center distance between a pair of mating gears with known tooth thicknesses, Eqs. (43) and (44) would be solved in the order given. As an example, let

$$\begin{array}{lcl} r_1 = 2.0000 \text{ in.} & & R_1 = 3.0000 \text{ in.} \\ t_1 = 0.2850 \text{ in.} & & T_1 = 0.2700 \text{ in.} \\ & \alpha_1 = 20 \text{ deg.} & \\ n = 24 \text{ teeth} & & N = 36 \text{ teeth} \\ & C_1 = 5.0000 \text{ in.} & \end{array}$$

From Eq. (43), we have

$$\text{inv}\,\alpha_2 = \frac{24(0.2850 + 0.2700) - 4\pi}{4(24 + 36)} + 0.014904 = 0.01804 \text{ radians.}$$

From the table of involute functions, we get

$$\alpha_2 = 21 \text{ deg. } 16 \text{ min. } 5 \text{ sec.}$$

whence,

$$\cos \alpha_2 = 0.93189$$

From Eq. (44), we get

$$C_2 = \frac{5 \times 0.93969}{0.93189} = 5.0418 \text{ in.}$$

PROBLEM 9.—*Given the tooth thickness of a pair of similar mating gears, to determine the center distance at which they will mesh tightly.*

This problem is a simplified example of the preceding one, and the same symbols and Fig. 36 will be used.

The necessary equations can be derived from the preceding Eqs. (43) and (44) by making the values on both gears identical. Whence, from Eq. (43), we get

$$\operatorname{inv} \alpha_2 = \frac{2(nt_1 - \pi r_1)}{4nr_1} + \operatorname{inv} \alpha_1 = \frac{nt_1 - \pi r_1}{2nr_1}$$

$$+ \operatorname{inv} \alpha_1 = \frac{t_1}{2r_1} - \frac{\pi}{2n} + \operatorname{inv} \alpha_1 \tag{45}$$

$$C_1 = 2r_1 \quad \text{and} \quad C_2 = 2r_2$$

$$C_2 = \frac{C_1 \cos \alpha_1}{\cos \alpha_2} \tag{46}$$

Thus, to determine the center distance between a pair of similar gears with known tooth thickness, Eqs. (45) and (46) would be solved in the order given. As an example, let

$$r_1 = 2.0000 \text{ in.}$$
$$t_1 = 0.2850 \text{ in.}$$
$$\alpha_1 = 20 \text{ deg.}$$
$$n = 24 \text{ teeth}$$
$$C_1 = 4.0000 \text{ in.}$$

From Eq. (45), we have

$$\operatorname{inv} \alpha_2 = \frac{0.2850}{4.0000} - \frac{3.1416}{48} + 0.014904 = 0.020704 \text{ radians.}$$

From the table of involute functions, we get

$$\alpha_2 = 22 \text{ deg. } 13 \text{ min. } 33 \text{ sec.}$$

whence,

$$\cos \alpha_2 = 0.92570$$

From Eq. (46), we get

$$C_2 = \frac{4 \times 0.93969}{0.92570} = 4.0604 \text{ in.}$$

PROBLEM 10.—*Given tooth thickness and pressure angle at the radius r_2, to determine the position of the meshing rack of known proportions and pressure angle.*

Referring to Fig. 37, let

r_2 = radius of gear at which tooth thickness is known
T_2 = tooth thickness at r_2
α_2 = pressure angle at r_2
α_1 = pressure angle of rack
r_1 = radius of gear where pressure angle is α_1
T_1 = tooth thickness at r_1
CP = circular pitch of rack
F = nominal addendum of rack, or distance from tip of rack tooth to point where thickness equals one-half the circular pitch
G = distance from center of gear to tip of rack tooth
N = number of teeth in gear

We know, from Eq. (39), that

$$r_1 = \frac{r_2 \cos \alpha_2}{\cos \alpha_1} \tag{47}$$

Also, from Eq. (40), that

$$T_1 = 2r_1\left(\frac{T_2}{2r_2} + \text{inv } \alpha_2 - \text{inv } \alpha_1\right)$$

We also know that the thickness of the tooth of the gear added to the thickness of the tooth of the rack where they mesh at the radius r_1 is equal to the circular pitch of the rack, whence,

$$\text{Thickness of rack tooth at } r_1 = CP - T_1.$$

Let x = distance of nominal pitch line on rack from r_1.

Then

$$x = \left[\frac{CP}{2} - (CP - T_1)\right]\frac{\cot \alpha_1}{2} = \left(T_1 - \frac{CP}{2}\right)\frac{\cot \alpha_1}{2}$$

whence,

$$G = r_1 - (F - x) = r_1 - F + x$$

Substituting the values of T_1 and x, we get

$$G = r_1 - F + \frac{\cot \alpha_1}{2}\left[2r_1\left(\frac{T_2}{2r_2} + \operatorname{inv} \alpha_2 - \operatorname{inv} \alpha_1\right) - \frac{CP}{2}\right] \quad (48)$$

Thus, to determine the position of a rack of one pressure angle, meshing with a gear whose tooth thickness is known at some other pressure angle, Eqs. (47) and (48) would be solved in the order given. As an example, let

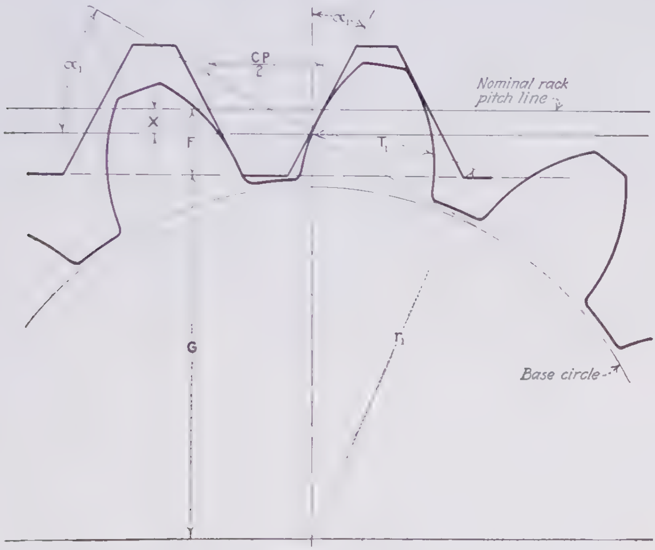

FIG. 37.—Determination of the position of the meshing rack with respect to a gear.

$$r_2 = 2.1250 \text{ in.}$$
$$T_2 = 0.1700 \text{ in.}$$
$$\alpha_2 = 27 \text{ deg. } 49 \text{ min. } 13 \text{ sec.}$$
$$\alpha_1 = 20 \text{ deg.}$$
$$CP = 0.5236 \text{ in.}$$
$$F = 0.1667 \text{ in.}$$

From Eq. (47), we get

$$r_1 = \frac{2.1250 \times 0.88441}{0.93969} = 2.0000 \text{ in.}$$

From Eq. (48), we get

$$G = 2.0000 - 0.1667 + 1.37375[4.0000(0.0400 + 0.042138 - 0.014904) - 0.2618] = 1.8431 \text{ in.}$$

Problem 11.—*Given the position of a rack of known proportions meshing with a gear, to determine the thickness of the tooth of the gear.*

This problem is the reverse of the preceding one. As we can determine the thickness of tooth at any radius once the thickness at some specific position is known, in this problem we will determine its thickness where its pressure angle is the same as that of the meshing rack.

Referring again to Fig. 37, we have

$$r_1 = \frac{N \cdot CP}{2\pi} \tag{49}$$

We also have, from the previous problem,

$$x = G + F - r_1$$

From Fig. 37, we see that

$$T_1 = 2x \tan \alpha_1 + \frac{CP}{2}$$

Substituting the value of x, we have

$$T_1 = 2 \tan \alpha_1 (G + F - r_1) + \frac{CP}{2} \tag{50}$$

Thus, to determine the thickness of the tooth of an involute gear when the position of a meshing rack is known, Eqs. (49) and (50) would be solved in the order given. As an example, let

$$\begin{aligned} \alpha_1 &= 20 \text{ deg.} \\ CP &= 0.5236 \text{ in.} \\ F &= 0.1667 \text{ in.} \\ N &= 24 \text{ teeth} \\ G &= 1.8600 \text{ in.} \end{aligned}$$

From Eq. (49), we get

$$r_1 = \frac{24 \times 0.5236}{2 \times 3.1416} = 2.0000 \text{ in.}$$

From Eq. (50), we get

$$T_1 = 2 \times 0.36397(1.8600 + 0.1667 - 2.0000) + 0.2618 = 0.2812 \text{ in.}$$

Problem 12.—*Given the center distance and pressure angle of a pair of meshing gears, to determine the positions of a meshing rack of different pressure angle.*

Let C_2 = center distance with pressure angle of α_2
N = number of teeth in gear
n = number of teeth in pinion
α_2 = pressure angle with center distance C_2
T_2 = tooth thickness of gear at C_2
t_2 = tooth thickness of pinion at C_2
α_1 = pressure angle of rack
C_1 = center distance with pressure angle of α_1
T_1 = tooth thickness of gear at C_1
t_1 = tooth thickness of pinion at C_1

The symbols CP, F, G, and g remain as before.

CP = circular pitch of rack
F = nominal addendum of rack
G = distance from center of gear to tip of rack tooth
g = distance from center of pinion to tip of rack tooth

We have, from Eq. (44),

$$C_1 = \frac{C_2 \cos \alpha_2}{\cos \alpha_1} \tag{51}$$

We know that

$$T_2 + t_2 = \frac{2\pi C}{N + n}$$

$$T_1 + t_1 = 2C_1\left(\frac{T_2 + t_2}{2C_2} + \operatorname{inv} \alpha_2 - \operatorname{inv} \alpha_1\right)$$

We also know that the sum of the thicknesses of the rack teeth at the meshing point is equal to

$$2CP - (T_1 + t_1)$$

Let x = sum of distance from meshing point to the nominal pitch line of rack, then

$$x = \{CP - [2CP - (T_1 + t_1)]\} \frac{\cot \alpha_1}{2} = (T_1 + t_1 - CP) \frac{\cot \alpha_1}{2}$$

whence,

$$(G + g) = C_1 + x - 2F$$

Substituting the values of x and $(T_1 + t_1)$, we get

$$(G + g) = C_1 - 2F + \frac{\cot \alpha_1}{2} \left[2C_1\left(\frac{\pi}{N + n} + \operatorname{inv} \alpha_2 - \operatorname{inv} \alpha_1\right) - CP\right] \tag{52}$$

Thus, to determine the sum of the rack positions on a pair of mating gears of different pressure angle than the rack, Eqs. (51) and (52) would be solved in the order given. As an example, let

$$\begin{aligned}
C_2 &= 5.1250 \text{ in.}\\
N &= 36 \text{ teeth}\\
n &= 24 \text{ teeth}\\
\alpha_2 &= 23 \text{ deg. } 32 \text{ min. } 30 \text{ sec.}\\
CP &= 0.5236 \text{ in.}\\
F &= 0.1667 \text{ in.}\\
\alpha_1 &= 20 \text{ deg.}
\end{aligned}$$

From Eq. (51), we get

$$C_1 = \frac{5.125 \times 0.91667}{0.93969} = 5.0000 \text{ in.}$$

From Eq. (52), we get

$$(G + g) = 5.0000 - 0.3334 + \frac{2.7475}{2}$$

$$\left[10\left(\frac{3.1416}{60} + 0.024798 - 0.014904\right) - 0.5236\right] = 4.8025 \text{ in.}$$

PROBLEM 13.—*Given the tooth thickness of a gear, to determine the position of a roll placed in the tooth space.*

Referring to Fig. 38, let

a_1 = radius of base circle
r_1 = radius at which tooth thickness is known
α_1 = pressure angle at r_1
T_1 = tooth thickness at r_1
W = radius of roll
r_2 = radius from center of gear to center of roll
α_2 = pressure angle at r_2
N = number of teeth in gear

The distance in circular measure from the center of the tooth to the center of the roll equals π/N.

The distance in circular measure from the center of the tooth to the origin of the involute on the base circle is equal to

$$\frac{T}{2r_1} + \text{inv}\,\alpha_1$$

Another involute is shown in the dotted line, which is generated from the same base circle as the involute of the tooth and which passes through the center of the roll. The distance from the center of the tooth to the origin of this second involute, in circular measure, is equal to

$$\frac{T}{2r_1} + \operatorname{inv} \alpha_1 + \frac{W}{a_1} \text{ approximately}$$

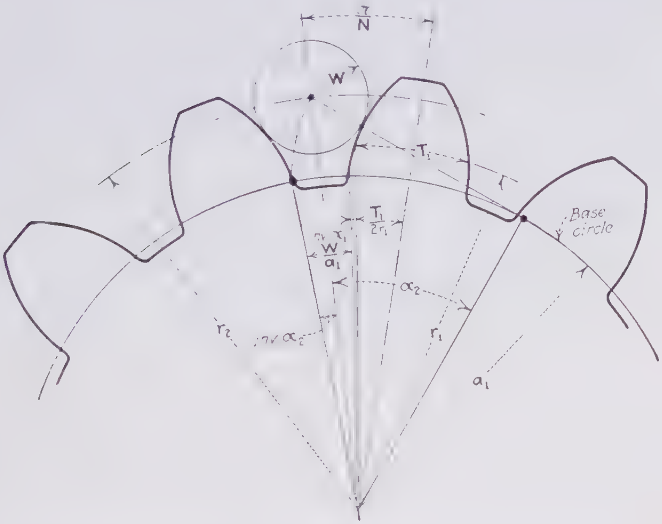

FIG. 38.—Determination of the position of a roll in the tooth space.

The distance in circular measure from the origin of this second involute to the radial line of the gear that passes through the center of the roll is the involute function of α_2, whence

$$\operatorname{inv} \alpha_2 = \frac{T_1}{2r_1} + \operatorname{inv} \alpha_1 + \frac{W}{a_1} - \frac{\pi}{N} \tag{53}$$

And from Eq. (42), we have

$$r_2 = \frac{r_1 \cos \alpha}{\cos \alpha_2} \tag{54}$$

Thus, to determine the position of a roll placed between the teeth of an involute gear, Eqs. (53) and (54) would be solved in the order given. As an example, let

$$r_1 = 2.0000 \text{ in.}$$
$$\alpha_1 = 20 \text{ deg.}$$
$$T_1 = 0.2618 \text{ in.}$$
$$W = 0.1500 \text{ in.}$$
$$N = 24 \text{ teeth.}$$

Then

$$a_1 = r \cos \alpha_1 = 2.0000 \times 0.93969$$

From Eq. (53), we have

$$\operatorname{inv} \alpha_2 = \frac{0.2618}{4.0000} + 0.014904 + \frac{0.1500}{2.0000 \times 0.93969} - \frac{3.1416}{24} = 0.029267$$

From the table of involute functions, we get

$$\alpha_2 = 24 \text{ deg. } 48 \text{ min. } 43 \text{ sec.}$$

From Eq. (54), we get

$$r_2 = \frac{2 \times 0.93969}{0.90769} = 2.0705 \text{ in.}$$

In the second chapter on the involute curve, many equations were given for determining various conditions on involute gears. The following are repeated here, with definite examples, so as to bring the more important equations needed to calculate involute tooth forms and characteristics together in one place.

PROBLEM 14.—*To determine the duration of contact between a pair of mating involute gears.*

When α = pressure angle
C = center distance
n = number of teeth in pinion
N = number of teeth in gear
E_1 = outside radius of pinion
E_2 = outside radius of gear
a_1 = radius of base circle of pinion
a_2 = radius of base circle of gear
r = pitch radius of pinion
R = pitch radius of gear
Pn = normal pitch

Referring to the previous chapter, from Eq. (16), we have

$$a_1 = r \cos \alpha \tag{55}$$
$$a_2 = R \cos \alpha \tag{56}$$

and

From Eq. (19), we have

$$Pn = \frac{2\pi a_1}{n} = \frac{2\pi a_2}{N} \tag{57}$$

And from Eq. (22), we have number of teeth in contact

$$= \frac{\sqrt{E_1^2 - a_1^2} + \sqrt{E_2^2 - a_2^2} - C \sin \alpha}{Pn} \tag{58}$$

Thus, to determine the duration of contact, or the number of teeth in contact, Eqs. (55), (56), (57), and (58) would be solved in the order given. As an example, let

$$\begin{aligned} \alpha &= 20 \text{ deg.} \\ C &= 7.0000 \text{ in.} \\ n &= 30 \text{ teeth} \\ N &= 40 \text{ teeth} \\ E_1 &= 3.2000 \text{ in.} \\ E_2 &= 4.2000 \text{ in.} \\ r &= 3.0000 \text{ in.} \\ R &= 4.0000 \text{ in.} \end{aligned}$$

From Eq. (55), we have

$$a_1 = 3.0000 \times 0.93969 = 2.8191 \text{ in.}$$

From Eq. (56), we have

$$a_2 = 4.0000 \times 0.93969 = 3.7588 \text{ in.}$$

From Eq. (57), we have

$$Pn = \frac{2 \times 3.1416 \times 2.8191}{30} = 0.5904 \text{ in.}$$

And from Eq. (58), we have

Number of teeth in contact =

$$\frac{\sqrt{(3.200)^2 - (2.8191)^2} + \sqrt{(4.200)^2 - (3.75876)^2} - 7 \times 0.34202}{0.5904}$$
$$= 1.70 \text{ teeth}$$

Problem 15.—*To determine the minimum radius of the bottom of a space without undercut.*

When

$$A = \text{undercut radius}$$
$$R = \text{radius of pitch circle}$$
$$\alpha = \text{pressure angle.}$$

From Eq. (31), we get, for a sharp-cornered rack,

$$A = R \cos^2 \alpha \tag{59}$$

If the rack has a radius or fillet at the point of the tooth, this rack can extend deeper by an amount equal to the height of this fillet without developing interference or undercut. Thus, when

$$f = \text{height of fillet at tip of rack tooth}$$
$$A = R \cos^2 \alpha - f$$

As an example, let

$$R = 1.0000 \text{ in.}$$
$$\alpha = 20 \text{ deg.}$$
$$f = 0.0320 \text{ in.}$$

From Eq. (60), we have

$$A = 1.0000 \times (0.93969)^2 - 0.0320 = 0.8510 \text{ in.}$$

PROBLEM 16.—*To determine the amount of involute profile removed when cutting below the undercut radius.*

When a rack-shaped cutter extends below the undercut radius, a portion of the involute profile is removed as the corner of the cutter moves out of engagement. To determine exactly the amount of the involute profile removed would involve very elaborate calculations. The following approximate method[1] gives results very close to the actual and slightly greater. It is sufficiently close, however, for all practical purposes.

When A = undercut radius
R = radius of pitch circle
a = radius of base circle
α = pressure angle
f = height of fillet at point of rack tooth
F = addendum of rack
e = excess depth of cutting, or depth below A

[1] By Ernest Wildhaber, of the Gleason Works, Rochester, New York.

x = radial height above the base circle of involute profile removed

Then

$$x = \frac{e^2}{6a \sin^2 \alpha} \text{ approximately} \tag{61}$$

As an example, let

$$R = 6.000 \text{ in.}$$
$$\alpha = 14\tfrac{1}{2} \text{ deg.}$$
$$F = 1.1570 \text{ in.}$$
$$f = 0.1570 \text{ in.}$$

From Eq. (60), we have

$$A = R \cos^2 \alpha - f = 5.4669 \text{ in.}$$

Depth of space below pitch circle

$$= 6.0000 - 1.1570 = 4.8350 \text{ in.}$$
$$e = 5.4669 - 4.8430 = 0.6239 \text{ in.}$$
$$a = R \cos \alpha = 6 \times 0.96815 = 5.8089$$

From Eq. (61), we have

$$x = \frac{e^2}{6a \sin^2 \alpha} = \frac{(0.6239)^2}{6(5.8089)(0.25038)^2} = 0.1781 \text{ in.}$$

The radius from the center of the gear to the beginning of the undestroyed involute profile is equal to $a + x$. In this example, it is equal to $5.80890 + 0.17813 = 5.98703$ in. In this example, there is only about 0.013 in. of useful profile below the pitch circle.

This equation can be simplified for use with definite pressure angles by substituting the corresponding values for the specified pressure angles.

For $14\frac{1}{2}$ deg., this would become

$$x = \frac{2.7448e^2}{R} \tag{62}$$

For 15 deg., this would become

$$x = \frac{2.5758e^2}{R} \tag{63}$$

For $17\frac{1}{2}$ deg., this would become

$$x = \frac{1.9327e^2}{R} \tag{64}$$

For 20 deg., this would become

$$x = \frac{1.5158e^2}{R} \tag{65}$$

For 22½ deg., this would become

$$x = \frac{1.2320e^2}{R} \tag{66}$$

CHAPTER IV

STANDARD GEAR-TOOTH FORMS

Before discussing in detail any of the standard gear-tooth systems or the considerations that led to their adoption, we will first define several of the terms which are commonly used.

GEAR TOOTH PARTS

Addendum.—The addendum of a gear tooth is the height of the gear tooth outside the pitch circle.

Dedendum.—The dedendum is the depth of the tooth space below the pitch circle.

Active Profile.—The active profile is that part of the gear-tooth profile which actually comes in contact with the profile of its mating tooth along the line of action.

Angle of Action.—The angle of action is the angle through which one tooth travels from the time it first makes contact with its mating tooth on the line of action until it ceases to be in contact.

Base Circle.—The base circle of an involute is the circle from which a line would be unwrapped to develop the involute curve.

Center Distance.—The center distance is the distance between the centers of a pair of mating gears.

Circular Pitch.—The circular pitch is the length of an arc of the pitch circle that corresponds to one tooth interval. It is equal to the circumference of the pitch circle divided by the number of teeth in the gear.

Clearance.—The clearance is the space provided between the top of the tooth of one gear and the bottom of its mating tooth space.

Diametral Pitch.—The diametral pitch is the ratio of the number of teeth to the pitch diameter of a gear. It represents the number of teeth per inch of pitch diameter. For example, if a gear of 2-in. pitch diameter has 12 teeth, its diametral pitch is equal to $^{12}\!/_{2}$, or 6.

Interference.—Interference relates to the existence of conditions that permit contact between mating teeth away from the line of action. Such conditions interfere with the transmission of uniform motion.

Line of Action.—The line of action is the line along which correct contact between mating teeth is made which results in the transmission of uniform motion from one gear to the other.

Module.—The module is the ratio of the pitch diameter of a gear with its number of teeth. It is the reciprocal of the diametral pitch. It represents the pitch diameter of a gear per tooth. For example, the module of a gear of 2-in. pitch diameter with 16 teeth would be $2/16$, or 0.125. The module multiplied by the number of teeth equals the pitch diameter.

Normal Pitch.—The normal pitch of an involute gear is the distance between two successive parallel involutes that form the profiles of two adjacent teeth. It is equal to the circumference of the base circle divided by the number of teeth in the gear.

Pitch Circle.—The pitch circle is the circle that represents a smooth disk that would transmit by friction the desired relative motion.

Pressure Angle.—The pressure angle of a pair of involute gears is the angle between the line of action and a line perpendicular to the common center-line of the two mating gears.

Whole Depth.—The whole depth is the total depth of the space on a gear measured radially between circles containing the tops of the teeth and the bottoms of the spaces.

Working Depth.—The working depth is the depth that the teeth of one gear extend into the spaces of its mating gear. It is equal to the sum of the addenda of mating gears. It is also equal to the whole depth minus the clearance.

Several of the foregoing terms are illustrated in Fig. 39.

Many factors are involved in the design of gear teeth. Here, as with all other engineering problems, the full advantage of one feature can seldom be taken without losing some of the advantages of other factors. The relative importance of the several factors must be weighed, and the final result is a compromise between the conflicting elements. As time goes on and as conditions change, the relative importance of different factors may also change, so that the compromise that was best for one set of conditions will be no longer justifiable for the new and changed conditions.

The primary purpose of gears is to transmit uniform motion. The first essential to attain this end consists in tooth profiles of conjugate form. Also, these profiles must be long enough so that the action is continuous; that is, the second pair of mating teeth must come in contact before the first pair have ceased to be in contact. In other words, the angle of action must be greater than the angle between successive teeth.

One word of caution at the outset: *No improvement in form alone will outweigh the importance of careful and painstaking workmanship in the production of the gears themselves.* In addition, the design of the mechanism in which the gears are used must

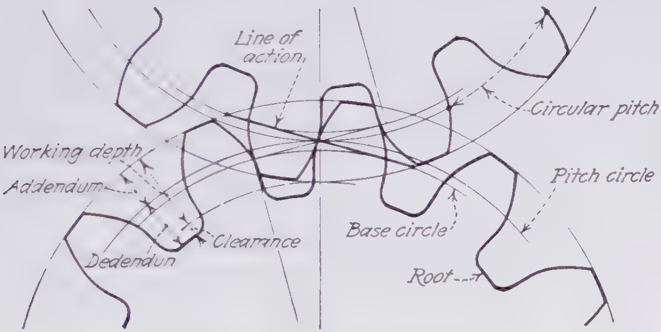

FIG. 39.—Gear-tooth nomenclature.

provide a sufficiently rigid support to hold the gears in proper alignment under their working loads.

The cut tooth gear is a comparatively modern achievement. Most of the earlier gear teeth were cast to form. These forms were shaped by the pattern maker to a more or less close approach to conjugate form. At the time when machining the gear teeth to form began to gain in favor, two forms were advocated—the cycloid and the involute. The cylcoid is a more sensitive form. It requires accurate center distances to operate properly and also requires a larger series of form cutters to cover a given range than the involute does, and so it has rapidly lost favor.

The biggest single step in advance in the early days of gear-cutting practice was the introduction of the 14½-deg. composite system. This system has been widely known as the "standard 14½-deg. involute system," although but a small part of the profile is of involute form. It is a compromise between the

TABLE II.—FORMULAS FOR 14½-DEG. COMPOSITE GEARS

$d.p.$ = diametral pitch
$c.p.$ = circular pitch
N = number of teeth in gear
D = outside diameter
D' = pitch diameter
s = addendum and also length of working face below the pitch line
$2s = D''$ = working depth of tooth
f = clearance
$s + f$ = dedendum
$D'' + f$ = whole depth
t = thickness of tooth on pitch line
C = center distance

Having	To get	Rule	Formula
Diametral pitch	Circular pitch	Divide 3.1416 by the diametral pitch	$c.p. = 3.1416/d.p.$
Pitch diameter and number of teeth	Circular pitch	Divide pitch diameter by the product of 0.3183 and number of teeth	$c.p. = D'/0.3183N$
Outside diameter and number of teeth	Circular pitch	Divide outside diameter by the product of 0.3183 and number of teeth plus two	$c.p. = D/0.3183(N + 2)$
Number of teeth and circular pitch	Pitch diameter	The continued product of the number of teeth, the circular pitch, and 0.3183	$D' = 0.3183N \times c.p.$
Number of teeth and outside diameter	Pitch diameter	Divide the product of number of teeth and outside diameter by number of teeth plus two	$D' = N \times D/(N + 2)$
Number of teeth and diametral pitch	Pitch diameter	Divide the number of teeth by the diametral pitch	$D' = N/d.p.$
Number of teeth and circular pitch	Outside diameter	The continued product of the number of teeth plus two, the circular pitch, and 0.3183	$D = (N + 2) \times 0.3183 \times c.p.$
Number of teeth and diametral pitch	Outside diameter	Divide the number of teeth plus two by the diametral pitch	$D = (N + 2)/d.p.$
Pitch diameter and diametral pitch	Outside diameter	Add two divided by the diametral pitch to the pitch diameter	$D = D' + 2/d.p.$
Pitch diameter and circular pitch	Number of teeth	Divide the product of pitch diameter and 3.1416 by the circular pitch	$N = 3.1416D'/c.p.$
Pitch diameter and diametral pitch	Number of teeth	Product of pitch diameter and diametral pitch	$N = D \times d.p.$
Circular pitch	Thickness of tooth	One-half the circular pitch	$t = c.p./2$
Diametral pitch	Thickness of tooth	Divide 1.5708 by the diametral pitch	$t = 1.5708/d.p.$
Circular pitch	Addendum	Multiply the circular pitch by 0.3183	$s = 0.3183c.p.$
Diametral pitch	Addendum	Divide one by the diametral pitch	$s = 1/d.p.$
Circular pitch	Dedendum	Multiply the circular pitch by 0.3683	$s + f = 0.3683c.p.$
Diametral pitch	Dedendum	Divide 1.157 by the diametral pitch	$s + f = 1.157/d.p.$
Circular pitch	Working depth	Multiply the circular pitch by 0.6366	$D'' = 0.6366c.p.$
Diametral pitch	Working depth	Divide two by the diametral pitch	$D'' = 2/d.p.$
Circular pitch	Whole depth	Multiply the circular pitch by 0.6866	$D'' + f = 0.6866c.p.$
Diametral pitch	Whole depth	Divide 2.157 by the diametral pitch	$D'' + f = 2.157/d.p.$
Circular pitch	Clearance	Multiply the circular pitch by 0.05	$f = 0.05c.p.$
Diametral pitch	Clearance	Divide 0.157 by the diametral pitch	$f = 0.157/d.p.$
Circular pitch	Diametral pitch	Divide 3.1416 by the circular pitch	$d.p. = 3.1416/c.p.$
Pitch diameter and number of teeth	Diametral pitch	Divide number of teeth by the pitch diameter	$d.p. = N/D'$
Outside diameter and number of teeth	Diametral pitch	Divide number of teeth plus two by outside diameter	$d.p. = (N + 2)/D$

NOTE: Table of formulas adapted from the "American Machinists' Handbook."
The center distance between any pair of these gears is found by adding together one-half of the pitch diameter of both gears.

TABLE III.—TOOTH PARTS OF 14½-DEG. COMPOSITE GEAR-TOOTH SYSTEM

Diametral pitch	Circular pitch	Thickness of tooth on pitch line	Addendum	Working depth of tooth	Depth of space below pitch line	Whole depth of tooth
½	6.2832	3.1416	2.0000	4.0000	2.3142	4.3142
¾	4.1888	2.0944	1.3333	2.6667	1.5428	2.8761
1	3.1416	1.5708	1.0000	2.0000	1.1571	2.1571
1¼	2.5133	1.2566	0.8000	1.6000	0.9257	1.7257
1½	2.0944	1.0472	0.6667	1.3333	0.7714	1.4381
1¾	1.7952	0.8976	0.5714	1.1429	0.6612	1.2326
2	1.5708	0.7854	0.5000	1.0000	0.5785	1.0785
2¼	1.3963	0.6981	0.4444	0.8889	0.5143	0.9587
2½	1.2566	0.6283	0.4000	0.8000	0.4628	0.8628
2¾	1.1424	0.5712	0.3636	0.7273	0.4208	0.7844
3	1.0472	0.5236	0.3333	0.6667	0.3857	0.7190
3½	0.8976	0.4488	0.2857	0.5714	0.3306	0.6163
4	0.7854	0.3927	0.2500	0.5000	0.2893	0.5393
5	0.6283	0.3142	0.2000	0.4000	0.2314	0.4314
6	0.5236	0.2618	0.1667	0.3333	0.1928	0.3595
7	0.4488	0.2244	0.1429	0.2857	0.1653	0.3081
8	0.3927	0.1963	0.1250	0.2500	0.1446	0.2696
9	0.3491	0.1745	0.1111	0.2222	0.1286	0.2397
10	0.3142	0.1571	0.1000	0.1818	0.1157	0.2157
12	0.2618	0.1309	0.0833	0.1667	0.0964	0.1798
14	0.2244	0.1122	0.0714	0.1429	0.0826	0.1541
16	0.1963	0.0982	0.0625	0.1250	0.0723	0.1348
18	0.1745	0.0873	0.0556	0.1111	0.0643	0.1198
20	0.1571	0.0785	0.0500	0.1000	0.0579	0.1079
24	0.1309	0.0654	0.0417	0.0833	0.0482	0.0898
28	0.1122	0.0561	0.0357	0.0714	0.0413	0.0770
32	0.0982	0.0491	0.0313	0.0625	0.0362	0.0674
36	0.0873	0.0436	0.0278	0.0556	0.0321	0.0599
40	0.0785	0.0393	0.0250	0.0500	0.0289	0.0539
48	0.0654	0.0327	0.0208	0.0417	0.0241	0.0449

involute and cycloid and was arrived at when both forms were widely used.

The angle of 14½ deg. was chosen as the pressure angle of the involute portion of this system because its sine was practically

equal to ¼ (sin 14½ deg. = 0.25038), and it made an easy angle for the millwright or pattern maker to lay out. This angle is a heritage from the cast gear-tooth design, but it remains today the most commonly used angle.

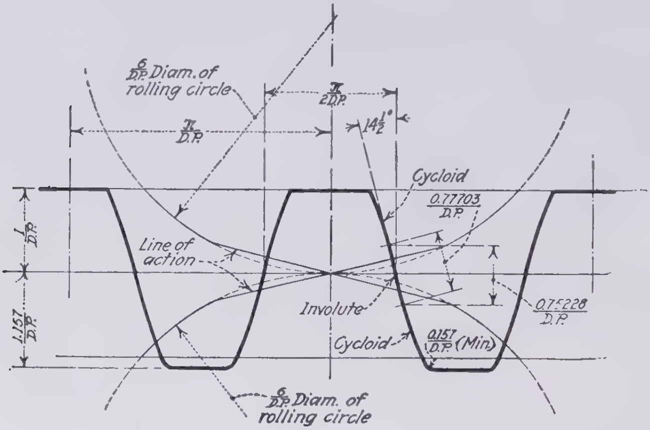

FIG. 40.—Basic rack for the 14½-deg. composite system of gear teeth.

The system is a fully interchangeable one, that is, any one gear will run with any other gear of the same pitch, regardless of tooth number, at a center distance which is directly proportional to the numbers of teeth in the gears. The system is based

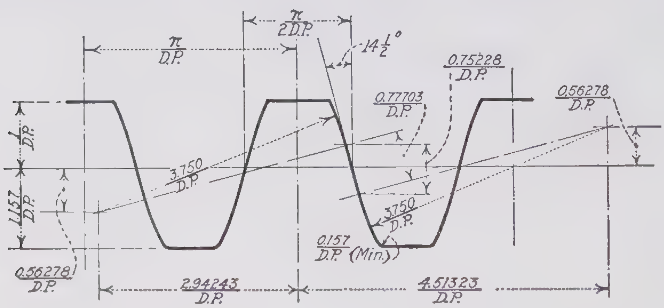

FIG. 41.—Approximation to the basic rack of the 14½-deg. composite system.

on a pinion of 12 teeth as the smallest of the series and extending up to a rack as the largest of the series of gear sizes.

As a pinion of 12 teeth of full involute form with a pressure angle of 14½ deg. would be badly undercut, the form is modified from the involute form to a cycloidal one which produces a radial line below the base circle of the 12-tooth pinion. In order to

keep the form of the basic rack of the system symmetrical, which is necessary to maintain an interchangeable system, the form of the addendum of the gears is also modified in the same manner. The result is the basic rack shown in Fig. 40.

The cycloidal form is a very difficult one to produce accurately. A very close approximation to the cycloid shown in Fig. 40 can be made with the arc of a circle, as shown in Fig. 41. The modification here is of segmental form and is a theoretically correct profile for the basic rack of a system of interchangeable conjugate gears. This is the basic-rack form actually used in practice.

This $14\frac{1}{2}$-deg. composite system has been adopted by the American Gear Manufacturers' Association and also by a sectional committee of the American Engineering Standards Committee as a tentative standard. The tooth form has the following proportions:

$d.p.$ = diametral pitch

M = module, the inverse of the diametral pitch

$$\text{Addendum} = \frac{1}{d.p.} = M$$

$$\text{Dedendum} = \frac{1.157}{d.p.} = 1.157M$$

$$\text{Working depth} = \frac{2}{d.p.} = 2M$$

$$\text{Whole depth} = \frac{2.157}{d.p.} = 2.157M$$

$$\text{Clearance} = \frac{0.157}{d.p.} = 0.157M.$$

In Table II, formulas are presented that give the general proportions of this system. In Table III, the chief dimensions have been calculated for the most commonly used diametral pitches.

THE FORM MILLING OF GEARS

The $14\frac{1}{2}$-deg. composite system was developed for formed milling cutters, but there is no reason why it cannot be produced by generating methods, if desired. When a generating method is employed, the generating cutter must take the form of the corresponding gear or of the basic rack of the system.

Form milling of gears is the most widely used method of producing gears today and probably always will be. This does not

mean that more gears are produced by form milling than by other methods, but rather that form milling is employed in more places than are other methods. This is because, by form milling, gears can be made without any other special equipment than the form-milling cutters. For general jobbing and repair work, a standard milling machine equipped with an indexing head or a dividing head is all that is required.

This method also lends itself readily to the production of gears in quantity. Standard gear-cutting machines are on the

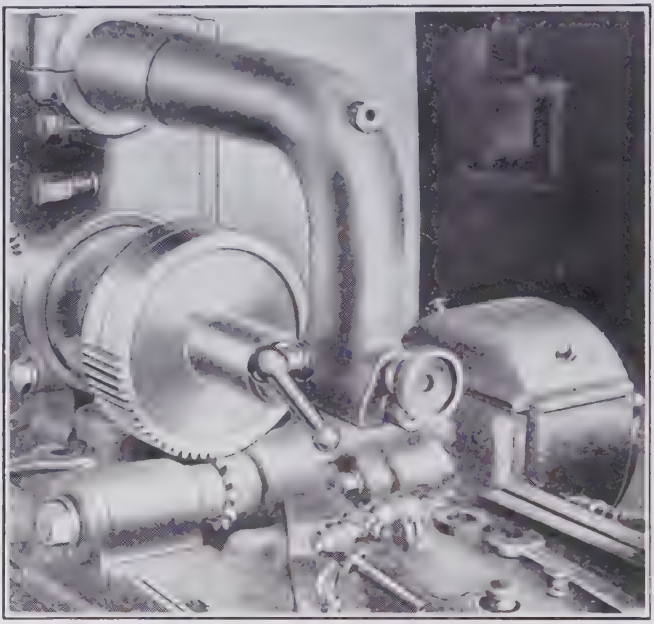

Fig. 42.—Cutting of a gear blank by form milling.

market for milling gears. These are made in many types. Some have single work arbors, while others have several and thus cut several gears or stacks of gears at the same time. All of these machines are automatic. Figure 42 shows a representative gear-cutting machine of this type with a single work arbor.

The milling cutters are form cutters relieved so that they can be sharpened by grinding the face of the cutter tooth without losing their form. Such a cutter is shown in Fig. 43.

Theoretically, a different cutter is required for each gear of a different number of teeth. A series of cutters has been devel-

oped, however, for the $14\frac{1}{2}$-deg. standard tooth forms so that only eight cutters are required for each pitch. These are adapted to cut from a pinion of 12 teeth to a rack and are numbered, respectively, 1, 2, 3, 4, etc.

No. 1 cutter will produce gears from 135 teeth to a rack.
No. 2 cutter will produce gears from 55 to 134 teeth.
No. 3 cutter will produce gears from 35 to 54 teeth.
No. 4 cutter will produce gears from 26 to 34 teeth.
No. 5 cutter will produce gears from 21 to 25 teeth.
No. 6 cutter will produce gears from 17 to 20 teeth.
No. 7 cutter will produce gears from 14 to 16 teeth.
No. 8 cutter will produce gears from 12 to 13 teeth.

Fig. 43.—Typical form-milling cutter for gear teeth.

These cutters are usually made of correct form for gears of the smallest number of teeth in their range. Thus, a No. 5 cutter is correct for 21 teeth and approximates very closely the form for the others in the range. These cutters are satisfactory where the conditions which the gears must meet are not severe. For more exacting gears, cutters of the half numbers listed above are made as follows:

No. 1½ cutter will produce gears from 80 to 134 teeth.
No. 2½ cutter will produce gears from 42 to 54 teeth.
No. 3½ cutter will produce gears from 30 to 34 teeth.
No. 4½ cutter will produce gears from 23 to 25 teeth.
No. 5½ cutter will produce gears from 19 to 20 teeth.
No. 6½ cutter will produce gears from 15 to 16 teeth.
No. 7½ cutter will produce gears with 13 teeth.

Special cutters are also made of correct form for any tooth numbers and any pressure angle and tooth form. This is an expense that is often justified for the production of large quantities of duplicate gears.

Great care must be exercised in setting up a milling machine to cut gears, if reasonably accurate results are desired. The center of the cutter must be accurately aligned with the center of the blank. For this reason, form cutters often have a line graved on them to locate the center of the cutter profile. The cutter must also be set closely to the proper depth. In fact, any mislocation from the correct relative positions of the cutter and gear blanks is a source of trouble.

Analysis of Milled Gears.[1]—As noted before, theoretically, a cutter, which is correct for a certain number of teeth, is incorrect for any other number. Practically, however, this error is slight. Furthermore, this theoretical error can be largely compensated for by a slight adjustment in the setting of the cutter.

Again, it often happens that a gear with 23 teeth, for example, is required, while the nearest cutter available is the No. 4 instead of the No. 5, which is ordinarily used. The No. 4 cutter can be used with good results by slightly modifying the depth of cut.

Furthermore, it is often desirable to provide more backlash on the gears than is present when the gears are milled to the theoretical depth. This is accomplished by sinking the cutters deeper into the blanks. Theoretically, this displaces the profile and alters the gear-tooth action. These errors also can be largely compensated for by a suitable adjustment in the depths of cut.

A study of the errors introduced by the foregoing factors will show that they can be reduced and compared to the errors in eccentric gears. The analysis, itself, is involved and requires much study to follow. It is omitted for several reasons. In the first place, the space available is not sufficient to make a clear

[1] Ernest Wildhaber, page 757, Vol. 59, *American Machinist*.

and complete exposition of the subject; in the second place, the majority of us are not so much interested in the derivation of the various equations as we are in the equations themselves and in their practical application; while, in the third place, the inclusion of the involved derivations would tend to obscure the simplicity of the final results.

Eccentric Gears.—An analysis will show that, if an eccentric involute gear, driving a rack, turns uniformly about its axis of rotation, the velocity of the rack will fluctuate according to a pure sine curve. This holds true for any amount of eccentricity.

If an eccentric gear, driving a concentric gear, turns uniformly about its axis of rotation, the velocity of the concentric gear will fluctuate almost according to a sine curve. When the amount of eccentricity is small, and we are dealing only with relatively small errors in this analysis, the variation from the sine curve is so slight that it may be neglected.

If an eccentric gear drives another eccentric gear, the velocity of the driven gear will fluctuate according to two sine curves. If the tooth numbers and the amount of eccentricity are alike on both gears, they may fully compensate for each other. When the tooth numbers are alike but the amount of eccentricity is different, they can be made to compensate partly for each other. In other words, the resulting error will be the same as if only one gear was eccentric with an error equal to the difference in eccentricity. When the tooth numbers are different, they cannot, as a rule, compensate for each other.

Purely eccentric gears, with small or large eccentricity, give a varying but continuous action. Each succeeding tooth takes over the load from the preceding one smoothly, without impact. This smoothness of action requires, of course, a sufficiently large angle of contact, a condition which is also necessary for smoothly running concentric gears. Furthermore, the tooth load should be large enough to keep the teeth of the two gears in contact.

Error When Milling Too Deep.—When a cutter is set to cut deeper than the theoretical depth, the tooth profiles of the cutter no longer correspond to the center of rotation of the blank; the center of the profile of each tooth space is then located beyond the actual center of the gear blank. This causes a somewhat different action than pure eccentricity, because, instead of having one center of eccentricity, each tooth space now has its own center of eccentricity. It is quite common in practice to cut too

deep. This procedure introduces the required play or backlash into the gears in order to avoid jamming. The effect of cutting too deep is to give a variable and a slower motion to the driven gear. Such a gear will move its rack too slowly. The following tooth comes into action too early and with a blow. This premature contact is also edge contact, as the tip edge of the rack does not contact tangentially with the flank of the pinion tooth but is somewhat tilted. The pressure is thus transmitted through the edge as long as the tip edge of the rack and the flank of the pinion tooth stand up. Such edge contact is dangerous and should be avoided.

The same relation as for a rack drive holds substantially true for gear drives, where one gear, which is cut too deep, meshes with another correctly cut gear. If the driver is cut too deep, then the driven gear has a tendency to fall back, and the contact on every tooth will start with edge contact. On the other hand, if the driven gear is cut too deep, while those of the driver are cut to correct depth, then the driven gear has the tendency to advance. This contact will also start with a blow but with one which is reduced as compared with the previous case because of the momentum of the moving parts. There is, moreover, no edge contact. The contact does not start at the bottom of the active face of the driving tooth and the tip of the driven tooth but starts somewhat above the bottom of the active profile of the driving tooth and below the tip of the driven tooth. It is evident that this condition is much more favorable than edge contact.

The influence of cutting deeper is dependent upon the number of teeth in the particular gears. Pinions with small tooth numbers are much more sensitive than gears with large tooth numbers. The amount of falling back or advancing of the driven gear is inversely proportional to the number of teeth in the gear which has been cut to a given excess depth. In other words, if, in different drives, one gear is always cut to an excess depth which is proportional to its tooth number, then all these drives will contain the same error.

The foregoing refers to the cutting of one gear of a drive too deep while its mating gear is cut to the correct depth, and to the using of cutters corresponding to the exact number of teeth in the gears. By cutting both gears of a drive too deep, the error may be partly or even entirely compensated for. If two

gears are cut with cutters that correspond to their exact tooth numbers but are cut to an excess depth proportional to their respective tooth numbers, the errors will then just compensate for each other.

Action of Gears Cut with Formed Cutters Corresponding to Different Tooth Numbers.—The milling of gears with different cutters than those which correspond to their exact tooth numbers has two effects. First, it produces a certain drop in the velocity of the driven gear from the beginning to the end of contact. Second, it tends to shift the period of contact ahead; that is, the contact will start at or near the bottom of the active profile on the driver but will end before it reaches the top of the driving tooth.

In such gears, only one tooth can be in contact at a time. In other words, the duration of contact of any one tooth lasts for one tooth interval and no longer. If the tooth profiles are long enough to provide this mesh, this drawback is not very important. If the tooth profiles are not long enough, however, edge contact will occur. The tooth profiles of the 14½-deg. composite system are sufficiently long to avoid this condition.

Improving Drives.—We have it in our power, however, to make a considerable improvement in the action of these gears by making one error, such as results from using a range cutter, compensate to a large extent for another, such as results from cutting the teeth deeper. In this way, we can force the contact toward the middle of the mesh and always avoid edge contact. At the same time, we can always introduce the desired amount of backlash safely.

It has been customary to cut a gear with a cutter corresponding to a smaller tooth number, if no exact tooth-number cutter is available. Many drives, however, can be greatly improved by using on one gear a cutter which corresponds to a larger tooth number, while the other gear is cut with the usual cutter.

The following simple equations must be solved to obtain these results:

When n = number of teeth in pinion
n' = exact tooth number of cutter for pinion
d = difference in tooth number for pinion, or $(n - n')$
N = number of teeth in gear
N' = exact tooth number of cutter for gear

D = difference in tooth number for gear, or $(N - N')$
D' = drop in velocity ratio during mesh
b = excess depth to cut teeth for tooth-number compensation
B = excess depth to give desired backlash
B' = total excess depth to cut teeth

Then

$$D' = \frac{D}{N \times N'} + \frac{d}{n \times n'} \tag{67}$$

When n' or N' is larger than N or N, respectively, the values of d or D will be negative. If the cutters are selected, one of larger and the other of smaller tooth number, so that Eq. (67) becomes equal to zero, true uniform-motion gears will be realized. The value of this equation should be kept as small as possible and always positive. Negative values must be avoided. Preferably, the smaller gear is milled with the cutter which corresponds to a larger tooth number. Equation (67) must be solved first to insure that a proper selection of cutters has been made. If the value is negative, or if it is large, another selection of cutters should be made.

The following equations are based on 14½-deg. gears of 1 d.p. For other diametral pitches, the results are divided by the required diametral pitch.

$$b = 0.2n'\left(\frac{D}{N \times N'} - \frac{d}{n \times n'}\right) \tag{68}$$

The value of b may be positive or negative. If the value of b is negative, it would mean that the pinion should be cut that amount shallower than standard. In these equations, the term "pinion" always refers to the driving gear, which is usually the smaller of the pair.

$$B = 2 \times \text{desired backlash} \tag{69}$$

The total excess depths to cut the gears, with due allowance for backlash and compensation for the cutters used, are given in the following equations:

For pinion n:

$$B' = B - (B - b)\frac{N'}{N' + n'} \tag{70}$$

For gear N:

$$B' = (B - b)\frac{N'}{N' + n'} \tag{71}$$

Equations (70) and (71) for the excess depths may be used, as long as both excess depths are positive; as noted before, b itself may be positive or negative.

For convenience, we will again list the series of eight range cutters with their exact tooth numbers.

Number of cutter	Exact tooth number	Number of cutter	Exact tooth number
1	135	5	21
2	55	6	17
3	35	7	14
4	26	8	12

As an example, we will take a 16-tooth pinion driving a 29-tooth gear, 6 d.p., to run with 0.010 in. backlash. This backlash would correspond to six times as much on a 1 d.p. gear, or 0.060 in. We will use a No. 6 cutter for the pinion and a No. 4 cutter for the gear. Thus, we have

$$\begin{aligned} n &= 16 \text{ teeth} \\ n' &= 17 \text{ teeth} \\ d &= -1 \text{ tooth} \\ N &= 29 \text{ teeth} \\ N' &= 26 \text{ teeth} \\ D &= 3 \text{ teeth} \end{aligned}$$

From Eq. (67), we get

$$D' = \frac{3}{29 \times 26} - \frac{1}{16 \times 17} = +0.000303$$

This value is small and positive and is therefore satisfactory. It is so near to zero that practically perfect action will be secured between the gear teeth.

From Eq. (68), we get

$$b = 0.2 \times 17\left(\frac{3}{29 \times 26} + \frac{1}{16 \times 17}\right) = 0.0260 \text{ in.}$$

From Eq. (69), we get

$$B = 2 \times 0.060 = 0.120 \text{ in.}$$

From Eq. (70), we get, for the excess depth to cut the 16-tooth pinion,

$$B' = 0.120 - (0.120 - 0.02603)\frac{26}{26 + 17} = 0.0632 \text{ in.}$$

Reduced to 6 d.p., this excess depth becomes 0.0105 in.

TABLE IV.—CORRECTED TOOTH DEPTH WITH VARIOUS COMBINATIONS OF RANGE-MILLING CUTTERS, 1-D.P. GEARS

Number of teeth in gear		Number of teeth in pinion																	
		12		13		14		15		16		17		18		19		20	
		Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches
12	Pinion	8	2.1870																
	Gear	8	2.1870																
13	Pinion	8	2.1947	8	2.1870														
	Gear	8	2.1793	8	2.1870														
14	Pinion	8	2.1847	8	2.1764	7	2.1870												
	Gear	7	2.1893	7	2.1976	7	2.1870												
15	Pinion	8	2.1901	8	2.1818	7	2.1937	7	2.1870										
	Gear	7	2.1839	7	2.1922	7	2.1803	7	2.1870										
16	Pinion	8	2.1950	7	2.2062	7	2.1995	7	2.1928	7	2.1870								
	Gear	7	2.1790	7	2.1688	7	2.1745	7	2.1812	7	2.1870								
17	Pinion	8	2.1818	8	2.1728	7	2.1819	7	2.1768	7	2.1704	6	2.1870						
	Gear	6	2.1922	6	2.2012	6	2.1921	6	2.1972	6	2.2036	6	2.1870						
18	Pinion	8	2.1864	8	2.1774	7	2.1873	7	2.1817	7	2.1809	6	2.1925	6	2.1870				
	Gear	6	2.1876	6	2.1966	6	2.1867	6	2.1923	6	2.1931	6	2.1815	6	2.1870				
19	Pinion	8	2.1906	7	2.2014	7	2.1920	7	2.1848	6	2.2037	6	2.1975	6	2.1919	6	2.1870		
	Gear	6	2.1834	6	2.1726	6	2.1820	6	2.1892	6	2.1703	6	2.1765	6	2.1821	6	2.1870		
20	Pinion	8	2.1943	7	2.2055	7	2.1963	6	2.2153	6	2.2082	6	2.2020	6	2.1732	6	2.1683	6	2.1870
	Gear	6	2.1797	6	2.1685	6	2.1777	6	2.1587	6	2.1658	6	2.1720	5	2.2008	5	2.2057	6	2.1870

21	Pinion	8 2.1788	8 2.1690	7 2.1810	7 2.1730	7 2.1660	6 2.1838	6 2.1777	6 2.1722	6 2.1673
	Gear	5 2.1952	5 2.2050	5 2.1930	5 2.2010	5 2.2080	5 2.1902	5 2.1963	5 2.2018	5 2.2067
22	Pinion	8 2.1821	8 2.1723	7 2.1846	7 2.1766	7 2.1696	6 2.1879	6 2.1817	6 2.1763	6 2.1714
	Gear	5 2.1919	5 2.2017	5 2.1894	5 2.1974	5 2.2044	5 2.1861	5 2.1923	5 2.1977	5 2.2026
23	Pinion	8 2.1851	8 2.1754	7 2.1880	7 2.1800	6 2.1985	6 2.1916	6 2.1855	6 2.1890	5 2.2007
	Gear	5 2.1889	5 2.1986	5 2.1860	5 2.1940	5 2.1755	5 2.1824	5 2.1885	5 2.1950	5 2.1733
24	Pinion	8 2.1879	7 2.1995	7 2.1910	7 2.1830	6 2.2020	6 2.1950	6 2.1889	5 2.2100	5 2.2045
	Gear	5 2.1861	5 2.1745	5 2.1830	5 2.1910	5 2.1720	5 2.1790	5 2.1851	5 2.1640	5 2.1695
25	Pinion	8 2.1905	7 2.2024	7 2.1938	7 2.1666	6 2.2050	6 2.1982	6 2.1686	5 2.2135	5 2.2080
	Gear	5 2.1835	5 2.1716	5 2.1802	4 2.2074	5 2.1690	5 2.1758	4 2.2054	5 2.1605	5 2.1660
26	Pinion	8 2.1759	8 2.1644	7 2.1780	7 2.1694	7 2.1617	6 2.1790	6 2.1719	6 2.1649	6 2.1599
	Gear	4 2.1981	4 2.2086	4 2.1960	4 2.2046	4 2.2123	4 2.1950	4 2.2021	4 2.2091	4 2.2141
27	Pinion	8 2.1783	8 2.1677	7 2.1806	7 2.1719	7 2.1643	6 2.1820	6 2.1750	6 2.1680	6 2.1630
	Gear	4 2.1957	4 2.2063	4 2.1934	4 2.2021	4 2.2097	4 2.1920	4 2.1990	4 2.2060	4 2.2110
28	Pinion	8 2.1805	8 2.1699	7 2.1830	7 2.1744	7 2.1668	6 2.1849	6 2.1778	6 2.1709	5 2.1957
	Gear	4 2.1935	4 2.2041	4 2.1910	4 2.1996	4 2.2072	4 2.1891	4 2.1962	4 2.2031	4 2.1733
29	Pinion	8 2.1825	8 2.1720	7 2.1852	7 2.1766	6 2.1950	6 2.1875	6 2.1805	6 2.1736	5 2.1986
	Gear	4 2.1915	4 2.2020	4 2.1888	4 2.1974	4 2.1668	4 2.1865	4 2.1935	4 2.2004	4 2.1754
30	Pinion	8 2.1844	7 2.1966	7 2.1874	7 2.1786	6 2.1879	6 2.1900	6 2.1829	5 2.2074	5 2.2012
	Gear	4 2.1896	4 2.1774	4 2.1866	4 2.1954	4 2.1761	4 2.1840	4 2.1911	4 2.1666	4 2.1728
31	Pinion	8 2.1861	7 2.1985	7 2.1893	7 2.1806	6 2.2006	6 2.1923	6 2.1853	5 2.2099	5 2.2037
	Gear	4 2.1879	4 2.1755	4 2.1847	4 2.1934	4 2.1737	4 2.1817	4 2.1887	4 2.1641	4 2.1703
32	Pinion	8 2.1878	7 2.2004	7 2.1911	7 2.1825	6 2.2024	6 2.1945	6 2.1630	5 2.2122	5 2.2061
	Gear	4 2.1862	4 2.1736	4 2.1829	4 2.1915	4 2.1716	4 2.1795	3 2.2110	4 2.1618	4 2.1679
33	Pinion	8 2.1894	7 2.2021	7 2.1928	6 2.2136	6 2.2044	6 2.1965	6 2.1652	5 2.2144	5 2.2083
	Gear	4 2.1846	4 2.1719	4 2.1812	4 2.1604	4 2.1696	4 2.1775	3 2.2088	4 2.1596	4 2.1657
34	Pinion	8 2.1908	7 2.2037	7 2.1944	6 2.2154	6 2.2063	6 2.1985	6 2.1672	5 2.2165	5 2.2104
	Gear	4 2.1832	4 2.1703	4 2.1796	4 2.1586	4 2.1677	4 2.1755	3 2.2068	4 2.1575	4 2.1636
35	Pinion	8 2.1723	8 2.1608	7 2.1741	7 2.1646	7 2.1570	6 2.1766	6 2.1691	6 2.1624	6 2.1570
	Gear	3 2.2017	3 2.2132	3 2.1999	3 2.2094	3 2.2170	3 2.1974	3 2.2049	3 2.2116	3 2.2170
36	Pinion	8 2.1737	8 2.1623	7 2.1757	7 2.1662	7 2.1579	6 2.1784	6 2.1710	6 2.1641	6 2.1582
	Gear	3 2.2003	3 2.2117	3 2.1983	3 2.2078	3 2.2161	3 2.1956	3 2.2030	3 2.2099	3 2.2158

TABLE IV.—CORRECTED TOOTH DEPTH WITH VARIOUS COMBINATIONS OF RANGE-MILLING CUTTERS, 1-D.P. GEARS (*Continued*)

Number of teeth in gear		Number of teeth in pinion																	
		12		13		14		15		16		17		18		19		20	
		Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches
37	Pinion	8	2.1751	8	2.1636	7	2.1772	7	2.1677	7	2.1594	6	2.1801	6	2.1726	6	2.1659	6	2.1599
	Gear	3	2.1989	3	2.2104	3	2.1968	3	2.2063	3	2.2146	3	2.1939	3	2.2014	3	2.2031	3	2.2141
38	Pinion	8	2.1763	8	2.1649	7	2.1786	7	2.1691	7	2.1609	6	2.1818	6	2.1743	6	2.1675	6	2.1616
	Gear	3	2.1977	3	2.2091	3	2.1954	3	2.2049	3	2.2132	3	2.1922	3	2.1997	3	2.2065	3	2.2124
39	Pinion	8	2.1775	8	2.1661	7	2.1809	7	2.1705	7	2.1621	6	2.1833	6	2.1758	6	2.1690	5	2.1934
	Gear	3	2.1965	3	2.2079	3	2.1940	3	2.2035	3	2.2119	3	2.1907	3	2.1982	3	2.2050	3	2.1806
40	Pinion	8	2.1787	8	2.1673	7	2.1813	7	2.1718	7	2.1633	6	2.1848	6	2.1773	6	2.1705	5	2.1951
	Gear	3	2.1953	3	2.2067	3	2.1927	3	2.2022	3	2.2106	3	2.1892	3	2.1967	3	2.2035	3	2.1780
41	Pinion	8	2.1798	8	2.1683	7	2.1825	7	2.1730	6	2.1946	6	2.1862	6	2.1787	6	2.1719	5	2.1967
	Gear	3	2.1942	3	2.2057	3	2.1915	3	2.2010	3	2.1794	3	2.1878	3	2.1953	3	2.2021	3	2.1773
42	Pinion	8	2.1808	8	2.1693	7	2.1836	7	3.1741	6	2.1959	5	2.1875	6	2.1800	6	2.1732	5	2.1982
	Gear	3	2.1932	3	2.2047	3	2.1904	3	2.1999	3	2.1781	3	2.1865	3	2.1940	3	2.2008	3	2.1758
43	Pinion	8	2.1819	7	2.1950	7	2.1848	7	2.1753	6	2.1972	6	2.1888	6	2.1813	5	2.2066	5	2.1997
	Gear	3	2.1921	3	2.1790	3	2.1892	3	2.1987	3	2.1768	3	2.1852	3	2.1927	3	2.1674	3	2.1743
44	Pinion	8	2.1827	7	2.1960	7	2.1858	7	2.1763	6	2.1984	6	2.1900	6	2.1825	5	2.2080	5	2.2011
	Gear	3	2.1913	3	2.1780	3	2.1882	3	2.1977	3	2.1756	3	2.1840	3	2.1915	3	2.1660	3	2.1729

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45	Pinion	8	2.1836	7	2.1971	7	2.1868	7	2.1774	6	2.1996	6	2.1912	6	2.1837	5	2.2093	5	2.2024
	Gear	3	2.1904	3	2.1769	3	2.1872	3	2.1966	3	2.1744	3	2.1828	3	2.1903	3	2.1647	3	2.1716
46	Pinion	8	2.1845	7	2.1980	7	2.1878	7	2.1783	6	2.2006	6	2.1922	6	2.1848	5	2.2106	5	2.2037
	Gear	3	2.1895	3	2.1760	3	2.1862	3	2.1957	3	2.1734	3	2.1818	3	2.1892	3	2.1634	3	2.1703
47	Pinion	8	2.1854	7	2.1989	7	2.1887	7	2.1792	6	2.2017	6	2.1933	6	2.1858	5	2.2118	5	2.2049
	Gear	3	2.1886	3	2.1751	3	2.1853	3	2.1948	3	2.1723	3	2.1807	3	2.1882	3	2.1622	3	2.1691
48	Pinion	8	2.1862	7	2.1999	7	2.1896	7	2.1801	6	2.2027	6	2.1943	6	2.1868	5	2.2130	5	2.2061
	Gear	3	2.1878	3	2.1741	3	2.1844	3	2.1939	3	2.1713	3	2.1797	3	2.1872	3	2.1619	3	2.1679
49	Pinion	8	2.1869	7	2.2006	7	2.1904	6	2.2137	6	2.2037	6	2.1953	6	2.1570	5	2.2141	5	2.2072
	Gear	3	2.1871	3	2.1734	3	2.1836	3	2.1608	3	2.1703	3	2.1787	2	2.2170	3	2.1599	3	2.1668
50	Pinion	8	2.1877	7	2.2015	7	2.1913	6	2.2142	6	2.2046	6	2.1962	6	2.1580	5	2.2151	5	2.2082
	Gear	3	2.1863	3	2.1725	3	2.1827	3	2.1598	3	2.1694	3	2.1778	2	2.2160	3	2.1589	3	2.1658
51	Pinion	8	2.1883	7	2.2023	7	2.1921	6	2.2150	6	2.2055	6	2.1971	6	2.1590	5	2.2162	5	2.2092
	Gear	3	2.1857	3	2.1717	3	2.1819	3	2.1590	3	2.1685	3	2.1769	2	2.2150	3	2.1578	3	2.1648
52	Pinion	8	2.1890	7	2.2030	7	2.1928	6	2.2159	6	2.2064	6	2.1980	6	2.1599	5	2.2170	5	2.2102
	Gear	3	2.1850	3	2.1710	3	2.1812	3	2.1581	3	2.1676	3	2.1760	2	2.2141	3	2.1570	3	2.1638
53	Pinion	8	2.1897	7	2.2037	7	2.1936	6	2.2167	6	2.2072	6	2.1988	6	2.1609	5	2.2170	5	2.2112
	Gear	3	2.1843	3	2.1703	3	2.1804	3	2.1573	3	2.1668	3	2.1752	2	2.2131	3	2.1570	3	2.1628
54	Pinion	8	2.1903	7	2.2044	7	2.1942	6	2.2170	6	2.2080	6	2.1996	6	2.1618	5	2.2170	5	2.2121
	Gear	3	2.1837	3	2.1696	3	2.1798	3	2.1570	3	2.1660	3	2.1744	2	2.2122	3	2.1570	3	2.1619
55	Pinion	8	2.1677	8	2.1570	7	2.1692	7	2.1586	7	2.1570	6	2.1712	6	2.1627	6	2.1570	6	2.1570
	Gear	2	2.2063	2	2.2170	2	2.2048	2	2.2154	2	2.2170	2	2.2028	2	2.2118	2	2.2170	2	2.2170
56	Pinion	8	2.1684	8	2.1570	7	2.1699	7	2.1593	7	2.1570	6	2.1720	6	2.1635	6	2.1570	6	2.1570
	Gear	2	2.2056	2	2.2170	2	2.2041	2	2.2147	2	2.2170	2	2.2020	2	2.2105	2	2.2170	2	2.2170
57	Pinion	8	2.1690	8	2.1570	7	2.1706	7	2.1600	7	2.1570	6	2.1728	6	2.1644	6	2.1570	6	2.1570
	Gear	2	2.2050	2	2.2170	2	2.2033	2	2.2140	2	2.2170	2	2.2012	2	2.2096	2	2.2170	2	2.2170
58	Pinion	8	2.1696	8	2.1570	7	2.1713	7	2.1606	7	2.1570	6	2.1736	6	2.1651	6	2.1575	6	2.1570
	Gear	2	2.2044	2	2.2170	2	2.2027	2	2.2134	2	2.2170	2	2.2004	2	2.2089	2	2.2165	2	2.2170
59	Pinion	8	2.1702	8	2.1576	7	2.1719	7	2.1613	7	2.1570	6	2.1744	6	2.1659	6	2.1583	6	2.1570
	Gear	2	2.2038	2	2.2164	2	2.2021	2	2.2127	2	2.2170	2	2.1996	2	2.2081	2	2.2157	2	2.2170
60	Pinion	8	2.1707	8	2.1581	7	2.1726	7	2.1619	7	2.1570	6	2.1751	6	2.1666	6	2.1590	6	2.1570
	Gear	2	2.2033	2	2.2159	2	2.2014	2	2.2121	2	2.2170	2	2.1989	2	2.2074	2	2.2150	2	2.2170

TABLE IV.—CORRECTED TOOTH DEPTH WITH VARIOUS COMBINATIONS OF RANGE-MILLING CUTTERS, 1-D.P. GEARS (*Continued*)

Number of teeth in gear		Number of teeth in pinion																	
		21		22		23		24		25		26		27		28		29	
		Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches
21	Pinion	5	2.1870																
	Gear	5	2.1870																
22	Pinion	5	2.1915	5	2.1870														
	Gear	5	2.1825	5	2.1870														
23	Pinion	5	2.1957	5	2.1911	5	2.1870												
	Gear	5	2.1783	5	2.1829	5	2.1870												
24	Pinion	5	2.1995	5	2.1949	5	2.1668	5	2.1870										
	Gear	5	2.1745	5	2.1791	4	2.2072	5	2.1870										
25	Pinion	5	2.2030	5	2.1752	5	2.1706	5	2.1664	5	2.1870								
	Gear	5	2.1710	4	2.1988	4	2.2034	4	2.2076	5	2.1870								
26	Pinion	5	2.1838	5	2.1788	5	2.1742	5	2.1700	5	2.1661	4	2.1870						
	Gear	4	2.1902	4	2.1952	4	2.1998	4	2.2040	4	2.2079	4	2.1870						
27	Pinion	5	2.1871	5	2.1821	5	2.1775	5	2.1733	5	2.1694	4	2.1907	4	2.1870				
	Gear	4	2.1869	4	2.1919	4	2.1965	4	2.2007	4	2.2046	4	2.1833	4	2.1870				
28	Pinion	5	2.1902	5	2.1852	5	2.1809	4	2.1764	4	2.1981	4	2.1941	4	2.1904	4	2.1870		
	Gear	4	2.1838	4	2.1888	4	2.1931	4	2.1976	4	2.1759	4	2.1799	4	2.1836	4	2.1870		
29	Pinion	5	2.1930	5	2.1880	5	2.1834	5	2.1774	4	2.2013	4	2.1973	4	2.1936	4	2.1902	4	2.1870
	Gear	4	2.1810	4	2.1860	4	2.1806	4	2.1966	4	2.1727	4	2.1767	4	2.1804	4	2.1838	4	2.1870

30	Pinion	5	2.1957	5	2.1907	5	2.1861	4	2.2088	4	2.2043	4	2.2003	4	2.1966	4	2.1932	4	2.1899
	Gear	4	2.1783	4	2.1833	4	2.1879	4	2.1652	4	2.1697	4	2.1737	4	2.1774	4	2.1808	4	2.1841
31	Pinion	5	2.1982	5	2.1932	5	2.1589	4	2.2114	4	2.2071	4	2.2031	4	2.1994	4	2.1959	4	2.1597
	Gear	4	2.1758	4	2.1808	3	2.2151	4	2.1626	4	2.1669	4	2.1709	4	2.1746	4	2.1781	3	2.2143
32	Pinion	5	2.2006	5	2.1955	5	2.1616	4	2.2141	4	2.2097	4	2.2057	4	2.2020	4	2.1664	4	2.1627
	Gear	4	2.1734	4	2.1785	3	2.2124	4	2.1599	4	2.1643	4	2.1683	4	2.1720	3	2.2076	3	2.2113
33	Pinion	5	2.2028	5	2.1694	5	2.1641	4	2.2165	4	2.2123	4	2.2082	4	2.2045	4	2.1692	4	2.1655
	Gear	4	2.1712	3	2.2046	3	2.2099	4	2.1575	4	2.1618	4	2.1658	4	2.1695	3	2.2018	3	2.2085
34	Pinion	5	2.2048	5	2.1716	5	2.1664	4	2.2170	5	2.1573	4	2.2105	4	2.1758	4	2.1718	4	2.1682
	Gear	4	2.1692	3	2.2024	3	2.2076	4	2.1570	3	2.2167	4	2.1635	3	2.1982	3	2.2022	3	2.2058
35	Pinion	5	2.1795	5	2.1738	5	2.1686	5	2.1639	5	2.1595	4	2.1826	4	2.1783	4	2.1744	4	2.1707
	Gear	3	2.1945	3	2.2002	3	2.2054	3	2.2101	3	2.2145	3	2.1914	3	2.1957	3	2.1996	3	2.2033
36	Pinion	5	2.1816	5	2.1759	5	2.1707	5	2.1659	5	2.1616	4	2.1849	4	2.1807	4	2.1767	4	2.1730
	Gear	3	2.1924	3	2.1981	3	2.2033	3	2.2081	3	2.2124	3	2.1891	3	2.1933	3	2.1973	3	2.2010
37	Pinion	5	2.1836	5	2.1779	5	2.1727	5	2.1679	4	2.1918	4	2.1872	4	2.1829	4	2.1790	4	2.1753
	Gear	3	2.1904	3	2.1961	3	2.2013	3	2.2061	3	2.1822	3	2.1868	3	2.1911	3	2.1950	3	2.1987
38	Pinion	5	2.1854	5	2.1797	5	2.1746	5	2.1698	4	2.1939	4	2.1893	4	2.1851	4	2.1811	4	2.1775
	Gear	3	2.1886	3	2.1943	3	2.1994	3	2.2042	3	2.1801	3	2.1847	3	2.1889	3	2.1929	3	2.1965
39	Pinion	5	2.1872	5	2.1815	5	2.1763	5	2.1716	4	2.1959	4	2.1913	4	2.1871	4	2.1831	4	2.1794
	Gear	3	2.1868	3	2.1925	3	2.1977	3	2.2024	3	2.1781	3	2.1827	3	2.1869	3	2.1909	3	2.1946
40	Pinion	5	2.1889	5	2.1832	5	2.1780	4	2.2028	4	2.1978	4	2.1932	4	2.1890	4	2.1850	4	2.1813
	Gear	3	2.1851	3	2.1908	3	2.1960	3	2.1712	3	2.1762	3	2.1808	3	2.1850	3	2.1890	3	2.1927
41	Pinion	5	2.1905	5	2.1848	5	2.1796	4	2.2046	4	2.1996	4	2.1950	4	2.1908	4	2.1868	4	2.1831
	Gear	3	2.1835	3	2.1892	3	2.1944	3	2.1694	3	2.1744	3	2.1790	3	2.1832	3	2.1872	3	2.1909
42	Pinion	5	2.1920	5	2.1863	5	2.1811	4	2.2063	4	2.2014	4	2.1968	4	2.1926	4	2.1886	4	2.1849
	Gear	3	2.1820	3	2.1877	3	2.1929	3	2.1677	3	2.1726	3	2.1772	3	2.1814	3	2.1854	3	2.1891
43	Pinion	5	2.1934	5	2.1878	5	2.1826	4	2.2080	4	2.2031	4	2.1985	4	2.1942	4	2.1903	4	2.1866
	Gear	3	2.1806	3	2.1862	3	2.1914	3	2.1660	3	2.1709	3	2.1755	3	2.1798	3	2.1837	3	2.1874
44	Pinion	5	2.1948	5	2.1892	5	2.1839	4	2.2096	4	2.2046	4	2.2000	4	2.1958	4	2.1918	4	2.1881
	Gear	3	2.1792	3	2.1848	3	2.1901	3	2.1644	3	2.1694	3	2.1740	3	2.1782	3	2.1822	3	2.1859
45	Pinion	5	2.1962	5	2.1905	5	2.1853	4	2.2111	4	2.2061	4	2.2015	4	2.1973	4	2.1933	4	2.1896
	Gear	3	2.1778	3	2.1835	3	2.1887	3	2.1629	3	2.1679	3	2.1725	3	2.1767	3	2.1807	3	2.1844

TABLE IV.—CORRECTED TOOTH DEPTH WITH VARIOUS COMBINATIONS OF RANGE-MILLING CUTTERS, 1-D.P. GEARS (*Continued*)

Number of teeth in gear		Number of teeth in pinion																	
		21		22		23		24		25		26		27		28		29	
		Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches
46	Pinion	5	2.1974	5	2.1917	5	2.1866	4	2.2125	4	2.2075	4	2.2029	4	2.1987	4	2.1947	4	2.1911
	Gear	3	2.1766	3	2.1823	3	2.1874	3	2.1615	3	2.1665	3	2.1711	3	2.1753	3	2.1793	3	2.1829
47	Pinion	5	2.1986	5	2.1929	5	2.1877	4	2.2139	4	2.2089	4	2.2043	4	2.2001	4	2.1961	4	2.1924
	Gear	3	2.1754	3	2.1811	3	2.1863	3	2.1601	3	2.1651	3	2.1697	3	2.1739	3	2.1779	3	2.1816
48	Pinion	5	2.1998	5	2.1941	5	2.1889	4	2.2152	4	2.2103	4	2.2056	4	2.2015	4	2.1572	4	2.1938
	Gear	3	2.1742	3	2.1799	3	2.1851	3	2.1588	3	2.1637	3	2.1684	3	2.1725	2	2.2168	3	2.1802
49	Pinion	5	2.2009	5	2.1952	5	2.1901	4	2.2165	4	2.2115	4	2.2069	4	2.2087	4	2.1587	4	2.1950
	Gear	3	2.1731	3	2.1788	3	2.1839	3	2.1575	3	2.1625	3	2.1671	3	2.1713	2	2.2153	3	2.1790
50	Pinion	5	2.2020	5	2.1615	5	2.1911	4	2.2170	4	2.2128	4	2.2082	4	2.2039	4	2.1601	4	2.1963
	Gear	3	2.1720	2	2.2125	3	2.1829	3	2.1570	3	2.1612	3	2.1658	3	2.1701	2	2.2139	3	2.1777
51	Pinion	5	2.2030	5	2.1627	5	2.1921	4	2.2170	4	2.2139	4	2.2093	4	2.2051	4	2.1615	4	2.1572
	Gear	3	2.1710	2	2.2113	3	2.1819	3	2.1570	3	2.1601	3	2.1647	3	2.1689	2	2.2125	2	2.2168
52	Pinion	5	2.2040	5	2.1638	5	2.1578	4	2.2170	4	2.2150	4	2.2105	4	2.1676	4	2.1628	4	2.1585
	Gear	3	2.1700	2	2.2102	2	2.2162	3	2.1570	3	2.1590	3	2.1635	2	2.2064	2	2.2112	2	2.2155
53	Pinion	5	2.2049	5	2.1650	5	2.1590	4	2.2170	4	2.2161	4	2.2115	4	2.1688	4	2.1641	4	2.1598
	Gear	3	2.1691	2	2.2090	2	2.2150	3	2.1570	3	2.1579	3	2.1625	2	2.2052	2	2.2099	2	2.2142
54	Pinion	5	2.2059	5	2.1660	5	2.1600	4	2.2170	4	2.2170	4	2.2126	4	2.1700	4	2.1653	4	2.1610
	Gear	3	2.1681	2	2.2080	2	2.2140	3	2.1570	3	2.1570	3	2.1614	2	2.2040	2	2.2087	2	2.2130

55	Pinion Gear	5 2.1736 2 2.2004	5 2.1670 2 2.2070	5 2.1610 2 2.2130	5 2.1570 2 2.2170	5 2.1570 2 2.2170	4 2.1763 2 2.1977	4 2.1712 2 2.2028	4 2.1666 2 2.2074	4 2.1622 2 2.2118
56	Pinion Gear	5 2.1745 2 2.1995	5 2.1680 2 2.2060	5 2.1620 2 2.2120	5 2.1570 2 2.2170	5 2.1570 2 2.2170	4 2.1774 2 2.1966	4 2.1724 2 2.2016	4 2.1677 2 2.2063	4 2.1634 2 2.2106
57	Pinion Gear	5 2.1755 2 2.1985	5 2.1689 2 2.2051	5 2.1629 2 2.2111	5 2.1574 2 2.2166	5 2.1570 2 2.2170	4 2.1785 2 2.1955	4 2.1735 2 2.2005	4 2.1688 2 2.2052	4 2.1644 2 2.2096
58	Pinion Gear	5 2.1764 2 2.1976	5 2.1699 2 2.2041	5 2.1639 2 2.2101	5 2.1584 2 2.2156	5 2.1570 2 2.2170	4 2.1796 2 2.1944	4 2.1746 2 2.1994	4 2.1699 2 2.2041	4 2.1655 2 2.2085
59	Pinion Gear	5 2.1773 2 2.1967	5 2.1708 2 2.2032	5 2.1647 2 2.2093	5 2.1592 2 2.2148	5 2.1570 2 2.2170	4 2.1806 2 2.1934	4 2.1756 2 2.1984	4 2.1709 2 2.2031	4 2.1666 2 2.2074
60	Pinion Gear	5 2.1782 2 2.1958	5 2.1716 2 2.2024	5 2.1656 2 2.2084	5 2.1601 2 2.2139	5 2.1570 2 2.2170	4 2.1816 2 2.1924	4 2.1766 2 2.1974	4 2.1719 2 2.2021	4 2.1676 2 2.2064
		30	31	32	33	34	35	36	37	38
30	Pinion Gear	4 2.1870 4 2.1870								
31	Pinion Gear	4 2.1570 3 2.2170	4 2.1870 4 2.1870							
32	Pinion Gear	4 2.1593 3 2.2147	4 2.1896 4 2.1844	4 2.1870 4 2.1870						
33	Pinion Gear	4 2.1621 3 2.2119	4 2.1589 3 2.2151	4 2.1570 3 2.2170	4 2.1870 4 2.1870					
34	Pinion Gear	4 2.1648 3 2.2092	4 2.1616 3 2.2124	4 2.1585 3 2.2155	4 2.1570 3 2.2170	4 2.1870 4 2.1870				
35	Pinion Gear	4 2.1673 3 2.2067	4 2.1641 3 2.2099	4 2.1610 3 2.2130	4 2.1582 3 2.2158	4 2.1570 3 2.2170	3 2.1870 3 2.1870			
36	Pinion Gear	4 2.1696 3 2.2044	4 2.1664 3 2.2076	4 2.1633 3 2.2107	4 3.1605 3 2.2135	4 2.1579 3 2.2161	3 2.1897 3 2.1843	3 2.1870 3 2.1870		
37	Pinion Gear	4 2.1718 3 2.2022	4 2.1687 3 2.2053	4 2.1656 3 2.2084	4 2.1627 3 2.2113	3 2.1953 3 2.1787	3 2.1923 3 2.1817	3 2.1896 3 2.1844	3 2.1870 3 2.1870	

TABLE IV.—CORRECTED TOOTH DEPTH WITH VARIOUS COMBINATIONS OF RANGE-MILLING CUTTERS, 1-D.P. GEARS (*Continued*)

Number of teeth in gear		Number of teeth in pinion																	
		30		31		32		33		34		35		36		37		38	
		Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches	Cutter No.	Corrected depth, inches
38	Pinion	4	2.1740	4	2.1708	4	2.1678	3	2.2009	3	2.1978	3	2.1949	3	2.1921	3	2.1895	3	2.1870
	Gear	3	2.2000	2	2.2032	3	2.2062	3	2.1731	3	2.1762	3	2.1791	3	2.1819	3	2.1845	3	2.1870
39	Pinion	4	2.1760	4	2.1728	3	2.2066	3	2.2033	2	2.2002	3	2.1972	3	2.1945	3	2.1918	3	2.1893
	Gear	3	2.1980	3	2.2012	3	2.1674	3	2.1707	3	2.1738	3	2.1768	3	2.1795	3	2.1822	3	2.1847
40	Pinion	4	2.1779	4	2.1747	3	2.2089	3	2.2055	3	2.2024	3	2.1995	3	2.1967	3	2.1941	3	2.1916
	Gear	3	2.1961	3	2.1993	3	2.1651	3	2.1685	3	2.1716	3	2.1745	3	2.1773	3	2.1799	3	2.1824
41	Pinion	4	2.1798	3	2.2145	3	2.2110	3	2.2077	3	2.2045	3	2.2016	3	2.1988	3	2.1962	3	2.1937
	Gear	3	2.1942	3	2.1595	3	2.1630	3	2.1663	3	2.1695	3	2.1724	3	2.1752	3	2.1778	3	2.1803
42	Pinion	4	2.1815	3	2.2166	3	2.2130	3	2.2097	3	2.2066	3	2.2036	3	2.2009	3	2.1982	3	2.1957
	Gear	3	2.1925	3	2.1574	3	2.1610	3	2.1643	3	2.1674	3	2.1704	3	2.1731	3	2.1758	3	2.1783
43	Pinion	4	2.1831	3	2.2170	3	2.2150	3	2.2117	3	2.2085	3	2.2056	3	2.2028	3	2.2002	3	2.1977
	Gear	3	2.1909	3	2.1570	3	2.1590	3	2.1623	3	2.1655	3	2.1684	3	2.1712	3	2.1738	3	2.1763
44	Pinion	4	2.1847	4	2.1815	3	2.2168	3	2.2135	3	2.2104	3	2.2074	3	2.2047	3	2.2020	3	2.1995
	Gear	3	2.1893	3	2.1925	3	2.1572	3	2.1605	3	2.1636	3	2.1666	3	2.1693	3	2.1720	3	2.1745
45	Pinion	4	2.1862	4	2.1830	4	2.1800	3	2.2153	3	2.2121	3	2.2092	3	2.2064	3	2.2038	3	2.2013
	Gear	3	2.1878	3	2.1910	3	2.1940	3	2.1587	3	2.1619	3	2.1648	3	2.1676	3	2.1702	3	2.1727
46	Pinion	4	2.1876	4	2.1845	4	2.1814	3	2.2170	3	2.2138	3	2.2109	3	2.2081	3	2.2055	3	2.2030
	Gear	3	2.1864	3	2.1895	3	2.1926	3	2.1570	3	2.1602	3	2.1631	3	2.1659	3	2.1685	3	2.1710

47	Pinion	4	2.1890	4	2.1858	4	2.1828	4	2.1800	3	2.2154	3	2.2125	3	2.2097	3	2.2071	3	2.2046
	Gear	3	2.1850	3	2.1882	3	2.1912	3	2.1940	3	2.1586	3	2.1615	3	2.1643	3	2.1669	3	2.1694
48	Pinion	4	2.1904	4	2.1872	4	2.1842	4	2.1813	3	2.2170	3	2.2141	3	2.2113	3	2.2087	3	2.2062
	Gear	3	2.1836	3	2.1868	3	2.1898	3	2.1927	3	2.1570	3	2.1599	3	2.1627	3	2.1653	3	2.1678
49	Pinion	4	2.1916	4	2.1884	4	2.1854	4	2.1826	4	2.1799	3	2.2155	3	2.2128	3	2.2101	3	2.1611
	Gear	3	2.1824	3	2.1856	3	2.1886	3	2.1914	3	2.1941	3	2.1585	3	2.1612	3	2.1639	2	2.2129
50	Pinion	4	2.1928	4	2.1896	4	2.1866	4	2.1838	4	2.1811	3	2.2170	3	2.2142	3	2.2116	3	2.1629
	Gear	3	2.1812	3	2.1844	3	2.1874	3	2.1902	3	2.1929	3	2.1570	3	2.1598	3	2.1624	2	2.2111
51	Pinion	4	2.1940	4	2.1908	4	2.1878	4	2.1850	4	2.1823	3	2.2170	3	2.2156	3	2.1676	3	2.1646
	Gear	3	2.1800	3	2.1832	3	2.1862	3	2.1890	3	2.1917	3	2.1570	3	2.1584	3	2.2064	2	2.2094
52	Pinion	4	2.1951	4	2.1919	4	2.1889	4	2.1861	4	2.1834	3	2.2170	3	2.2170	3	2.1693	3	2.1662
	Gear	3	2.1789	3	2.1821	3	2.1851	3	2.1879	3	2.1906	3	2.1570	3	2.1570	2	2.2047	2	2.2078
53	Pinion	4	2.1962	4	2.1930	4	2.1900	4	2.1872	4	2.1845	3	2.2170	3	2.1740	3	2.1708	3	2.1677
	Gear	3	2.1778	3	2.1810	3	2.1840	3	2.1868	3	2.1895	3	2.1570	2	2.2000	2	2.2032	2	2.2063
54	Pinion	4	2.1570	4	2.1940	4	2.1911	4	2.1883	4	2.1856	3	2.2170	3	2.1755	3	2.1723	3	2.1692
	Gear	2	2.2170	3	2.1800	3	2.1829	3	2.1857	3	2.1884	3	2.1570	2	2.1985	2	2.2017	2	2.2048
55	Pinion	4	2.1581	4	2.1570	4	2.1570	4	2.1570	4	2.1570	3	2.1803	3	2.1770	3	2.1737	3	2.1707
	Gear	2	2.2159	2	2.2170	2	2.2170	2	2.2170	2	2.2170	2	2.1937	2	2.1970	2	2.2003	2	2.2033
56	Pinion	4	2.1593	4	2.1570	4	2.1570	4	2.1570	4	2.1570	3	2.1817	3	2.1783	3	2.1751	3	2.1720
	Gear	2	2.2147	2	2.2170	2	2.2170	2	2.2170	2	2.2170	2	2.1923	2	2.1957	2	2.1989	2	2.2020
57	Pinion	4	2.1604	4	2.1570	4	2.1570	4	2.1570	4	2.1570	3	2.1831	3	2.1797	3	2.1765	3	2.1734
	Gear	2	2.2136	2	2.2170	2	2.2170	2	2.2170	2	2.2170	2	2.1909	2	2.1943	2	2.1975	2	2.2006
58	Pinion	4	2.1615	4	2.1577	4	2.1570	4	2.1570	3	2.1880	3	2.1844	3	2.1810	3	2.1778	3	2.1747
	Gear	2	2.2125	2	2.2163	2	2.2170	2	2.2170	2	2.1860	2	2.1896	2	2.1930	2	2.1962	2	2.1993
59	Pinion	4	2.1625	4	2.1587	4	2.1570	4	2.1570	3	2.1892	3	2.1856	3	2.1822	3	2.1790	3	2.1759
	Gear	2	2.2115	2	2.2153	2	2.2170	2	2.2170	2	2.1848	2	2.1884	2	2.1918	2	2.1950	2	2.1981
60	Pinion	4	2.1635	4	2.1598	4	2.1570	4	2.1570	3	2.1904	3	2.1868	3	2.1835	3	2.1803	3	2.1772
	Gear	2	2.2105	2	2.2142	2	2.2170	2	2.2170	2	2.1836	2	2.1872	2	2.1905	2	2.1937	2	2.1968

From Eq. (71), we get, for the excess depth to cut the 29-tooth gear,

$$B' = (0.120 - 0.02603)\frac{26}{26 + 17} = 0.0568 \text{ in.}$$

Reduced to 6 d.p., this excess depth becomes 0.0095 in.

Ordinarily, this 16-tooth pinion would be cut with a No. 7 cutter, and the excess depth for backlash would be evenly divided between the two gears. Even with the ordinary cutter, an improvement in the action would be secured by adjusting the excess depth for backlash. In this case, we should have

$$n' = 14 \text{ teeth}$$
$$d = 2 \text{ teeth}$$

All other values are unchanged. Proceeding as before, from Eq. (67), we get

$$D' = \frac{3}{29 \times 26} + \frac{2}{16 \times 14} = 0.0129$$

The value of D' is a measure of the variation in velocity during one tooth movement of the gears. This last value of D' is over 40 times greater than the first one. The first solution, therefore, will produce a much better-running pair of gears than the second one. The latter can be greatly improved, nevertheless, by proper distribution of the excess depth of cut.

From Eq. (68), we get

$$b = 0.2 \times 14\left(\frac{3}{29 \times 26} - \frac{2}{16 \times 14}\right) = -0.0139 \text{ in.}$$

From Eq. (70), we get, for the excess depth to cut the 16-tooth pinion,

$$B' = 0.120 - (0.120 + 0.01386)\frac{26}{26 + 14} = 0.0330$$

Reduced to 6 d.p., this excess depth becomes 0.0055 in.

From Eq. (71), we get, for the excess depth to cut the 29-tooth gear,

$$B' = (0.120 + 0.01386)\frac{26}{26 + 14} = 0.0870$$

Reduced to 6 d.p., this excess depth becomes 0.0145 in.

These examples should be sufficient to indicate the use of the foregoing equations. One word of caution, however, should be given: When the tooth numbers of the cutters are both smaller

than the tooth numbers of the gears, edge contact seldom occurs, whether or not a correction is made for the depth of cut. On the other hand, if the tooth number of the cutter is larger than that of the gear, edge contact will occur unless a correction is made in the depth of the cut, as outlined above.

One important gear problem is how to avoid edge contact at the beginning of mesh. The tip edge of the driven tooth is a particularly dangerous edge to make contact with the flank of the driving tooth. When this danger exists in milled gears for any reason, it may always be avoided by cutting the driven gear a trifle deeper.

A table has been prepared which shows what cutter to use and the corrected depth of cut in order to employ the range-milling cutters to best advantage. This is a table of 1-d.p. gears with 0.030 in. backlash. For any other diametral pitch, the tabulated depths of cut should be divided by the desired diametral pitch. The first figure shows the cutter number to use for the pinion, or smallest gear; the second figure gives the corrected depth to cut the pinion; the third figure gives the cutter number to use for the gear; while the last figure gives the corrected depth of cut for the gear. These depths of cut are based on gear blanks of correct standard outside diameters.

As an example of the use of these tables of cutters and tooth depths, we will take a pair of 6-d.p. gears with 16 and 48 teeth. The outside diameters are calculated in the usual way. We have as the formula for the outside diameter D,

$$D = \frac{N + 2}{d.p.}$$

For 16 teeth, we have

$$D = \frac{16 + 2}{6} = 3.0000 \text{ in.}$$

For 48 teeth, we have

$$D = \frac{48 + 2}{6} = 8.3333 \text{ in.}$$

From the table of cutters and tooth depths for the 16–48 combination, we have

Number of teeth................	16	48
Cutter number..................	6	3
Tooth depth (for 1 d.p.), inches...	2.2027	2.1713

Dividing the 1-d.p. tooth depth by six we get the following:

6-d.p. tooth depths, inches	0.3654	0.3619

The final specifications for these gears would be

Number of teeth	16	48
Outside diameter, inches	3.0000	8.3333
Cutter	6-d.p. No. 6	6-d.p. No. 3
Depth to mill, inches	0.3654	0.3619
Backlash $\left(\frac{0.030}{6}\right)$, inches	0.0050	

With these, or any other milled gears, if there should be edge contact due to errors in cutting or errors in the cutters themselves, the remedy is to cut the driven gear a trifle deeper.

THE 14½-DEG. GENERATED GEAR-TOOTH SYSTEM

In the past, the different gear-tooth systems have been developed primarily around the method employed to produce them. Thus, with the introduction of generating or molding processes, such as hobbing and shaping, new gear-tooth systems came into existence. These new systems were influenced at the start by the existing practices. Later, they departed from them to a lesser

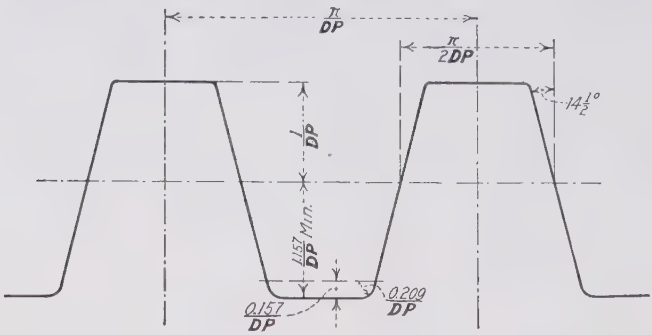


FIG. 44.—Basic rack of the 14½-deg. generated system.

or to a greater extent in order to obtain greater benefit of the new production processes.

For example, when hobbing was first introduced, the tooth design followed very closely the tooth design of milled gears, which, in their turn, had been largely influenced by the design of the teeth of cast gears. The tooth proportions and pressure angle for the hobbed gears remained the same as those for milled gears. The worm-shaped cutter represented the basic rack of the

gear system, and because the straight-sided, involute rack was the simplest form to reproduce on these cutters, this form was adopted. This method of generation introduced another 14½-deg. gear-tooth system with a basic rack of the form shown in Fig. 44. This 14½-deg. generated form will not mesh correctly with the 14½-deg. composite form used for milled gears. It is an entirely distinct gear-tooth system.

Analysis of 14½-deg. Generated Gear System.—When the numbers of teeth in the gears are large enough, this system gives excellent results. When the tooth numbers are small, however, excessive undercutting occurs. The following analysis of this gear-tooth system should make evident its limitations when small numbers of teeth are involved.

Most gear-tooth systems are based on a pinion of 12 teeth as the smallest gear of the system. We will therefore examine first the 12-tooth pinion of this system.

When E = outside radius
R = pitch radius
a = radius of base circle
F = addendum of basic rack, including clearance fillet
f = clearance on generating rack or hob
A = minimum root radius without undercut
H = root radius
α = pressure angle

We have the following values for a 12-tooth pinion of 1 d.p.:

$$E = 7.0000 \text{ in.}$$
$$R = 6.0000 \text{ in.}$$
$$H = 4.8430 \text{ in.}$$
$$F = 1.1570 \text{ in.}$$
$$f = 0.1570 \text{ in.}$$
$$\alpha = 14\tfrac{1}{2} \text{ deg.}$$

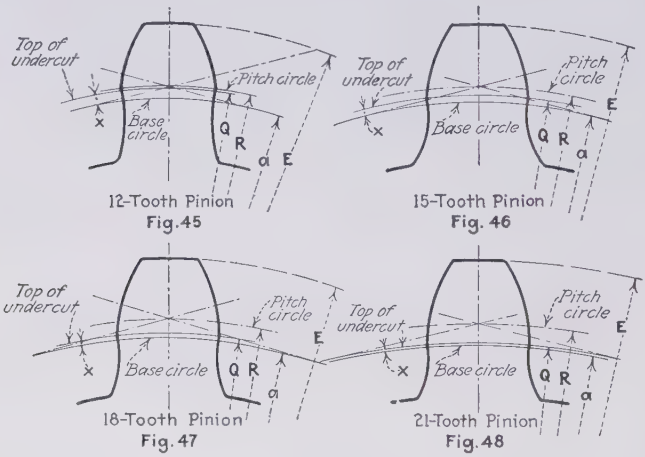
$a = R \cos \alpha = 5.8089$ (see Eq. (55))
$A = R \cos^2 \alpha - f = 5.4669$ (see Eq. (60))
e = excess depth beyond undercut limit $= A - H = 0.6239$.
x = radial height above the base circle of undercut part of profile.

$$x = \frac{2.7448e^2}{R} = 0.1781 \text{ (see Eq. (62))}$$

The shape of this pinion tooth is shown in Fig. 45. It will be noted that almost all of the involute profile below the pitch circle has been removed. If this pinion engaged with a rack

(a condition which would give the maximum contact possible). the amount of contact would be equal to the quotient of the length of the line of action, intercepted between the outside circle of the pinion and the circle through the top of the undercut, divided by the normal pitch. The result would be expressed in terms of tooth intervals.

The length of the line of action from the base circle to the outside circle equals $\sqrt{E^2 - a^2}$ (see Eq. (57))



FIGS. 45, 46, 47 and 48.—Tooth forms of pinions of 12, 15, 18 and 21 teeth, respectively, of the 14½-deg. generated system.

Let Q = radius to top of undercut, then

$$Q = a + x$$

and the length of the line of action from the base circle to the circle through the top of the undercut $= \sqrt{Q^2 - a^2}$

The length of the line of action, therefore, intercepted between the outside circle and the circle at the top of the undercut

$$= \sqrt{E^2 - a^2} - \sqrt{Q^2 - a^2}$$

This length, divided by the normal pitch Pn, gives us the measure of the maximum tooth contact possible.

$$Pn = \frac{2\pi a}{N}, \quad \text{where } N = \text{number of teeth (see Eq. (57))}$$

Whence, involute contact between a rack and an undercut gear

$$= \frac{\sqrt{E^2 - a^2} - \sqrt{Q^2 - a^2}}{Pn}$$

In this example,

$$\sqrt{E^2 - a^2} = 3.9060 \text{ in.}$$
$$Q = 5.8089 + 0.1781 = 5.9870 \text{ in.}$$
$$\sqrt{Q^2 - a^2} = 1.4503 \text{ in.}$$
$$Pn = 3.0415 \text{ in.}$$

Whence, contact with a rack = 0.807 tooth interval.

The length of contact on this pinion is not sufficient to give an overlap even when meshing with a rack. This contact becomes less where gears are engaged with it and becomes smaller as the number of teeth decreases. Where two of these 12-tooth pinions mesh together, the contact becomes equal to

$$\frac{2(R \sin \alpha - \sqrt{Q^2 - a^2})}{Pn} = \frac{0.1040}{3.0415} = 0.034 \text{ tooth interval}$$

It is evident that such a pinion has very little value as a means of transmitting power smoothly.

Theoretically, the length of the line of action must be equal to the normal pitch, or longer, in order that the motion may be transferred smoothly from one tooth to the next. In practice, it has been found that with a contact less than 1.40 tooth intervals, great care must be exercised in the production of such gears to secure smooth and quiet running. For the quiet transmission of any appreciable amount of power, contact of 1.40 tooth intervals or better should be secured, if possible. As small a contact as 1.20 intervals is used in extreme cases, but this requires extreme accuracy in the gears to secure smooth and quiet running.

The accompanying table gives values for 1-d.p. gears of small tooth numbers, made to the 14½-deg. generated gear system, which establish the amount of undercut and contact.

Figures 46, 47, and 48 show the forms of 15-, 18-, and 21-tooth gears of this system. Note how the top of the undercut recedes from the pitch circle toward the base circle as the number of teeth is increased.

A second table is given which shows the amount of contact that exists between gears of small tooth numbers made to this system. The values are expressed in terms of tooth intervals.

It will be seen from this table of contact that two 23-tooth pinions are the smallest equal pair that give a contact of 1.40

TABLE V.—14½-DEG. GENERATED SYSTEM OF GEAR TEETH

Number of teeth N	Radius of pitch circle R	Radius of base circle a	Undercut radius A	Root radius H	Excess depth of cut e	Height of undercut x	Radius to top of undercut Q	Outside radius E	$\sqrt{E^2 - a^2}$	$\sqrt{Q^2 - a^2}$	$R \sin \alpha$
12	6.00	5.80890	5.46686	4.8430	0.62386	0.17813	5.98703	7.00	3.9060	1.4503	1.5023
13	6.50	6.29298	5.93552	5.3430	0.59252	0.14825	6.44123	7.50	4.0802	1.3748	1.6275
14	7.00	6.77705	6.40417	5.8430	0.56117	0.12348	6.90053	8.00	4.2511	1.2996	1.7527
15	7.50	7.26113	6.87283	6.3430	0.52983	0.10274	7.36387	8.50	4.4188	1.2259	1.8779
16	8.00	7.74520	7.34148	6.8430	0.49848	0.08525	7.83045	9.00	4.5839	1.1524	2.0030
17	8.50	8.22928	7.81014	7.3430	0.46714	0.07047	8.29975	9.50	4.7475	1.0794	2.1282
18	9.00	8.71335	8.27879	7.8430	0.43579	0.05792	8.77127	10.00	4.9060	1.0065	2.2534
19	9.50	9.19743	8.74745	8.3430	0.40445	0.04726	9.24469	10.50	5.0653	0.9338	2.3786
20	10.00	9.68150	9.21610	8.8430	0.37310	0.03821	9.71971	11.00	5.2220	0.8612	2.5038
21	10.50	10.16558	9.68476	9.3430	0.34176	0.03057	10.19615	11.50	5.3769	0.7887	2.6290
22	11.00	10.64965	10.15341	9.8430	0.31041	0.02404	10.67369	12.00	5.5304	0.7163	2.7542
23	11.50	11.13373	10.62207	10.3430	0.27907	0.01859	11.15232	12.50	5.6825	0.6441	2.8794
24	12.00	11.61780	11.09072	10.8430	0.24772	0.01404	11.63184	13.00	5.8333	0.5719	3.0046
25	12.50	12.10188	11.55938	11.3430	0.21638	0.01028	12.11216	13.50	5.9829	0.4997	3.1298
26	13.00	12.58595	12.02803	11.8430	0.18503	0.00723	12.59318	14.00	6.1314	0.4275	3.2549
27	13.50	13.07003	12.49669	12.3430	0.15369	0.00480	13.07483	14.50	6.2789	0.3554	3.3801
28	14.00	13.55410	12.96534	12.8430	0.12234	0.00293	13.55703	15.00	6.4255	0.2833	3.5053
29	14.50	14.03818	13.43400	13.3430	0.09100	0.00156	14.03974	15.50	6.5712	0.2188	3.6305
30	15.00	14.52225	13.90265	13.8430	0.05965	0.00065	14.52290	16.00	6.7160	0.1409	3.7557
31	15.50	15.00633	14.37131	14.3430	0.02831	0.00014	15.00647	16.50	6.8601	0.0700	3.8809
32	16.00	15.49040	14.83996	14.8430	0.00000	0.00000	....	17.00	7.0035	0.0000	4.0061
33	16.50	15.97448	15.30862	15.3430	....	....	....	17.50	7.1461	......	4.1313
34	17.00	16.45855	15.77727	15.8430	....	....	....	18.00	7.2281	......	4.2565
35	17.50	16.94263	16.24593	16.3430	....	....	....	18.50	7.4295	......	4.3817
36	18.00	17.42670	16.71458	16.8430	....	....	....	19.00	7.5704	......	4.5068
37	18.50	17.91078	17.18324	17.3430	....	....	....	19.50	7.7107	......	4.6320
38	19.00	18.39485	17.65189	17.8430	....	....	....	20.00	7.8505	......	4.7572
39	19.50	18.87893	18.12055	18.3430	....	....	....	20.50	7.9898	......	4.8824
40	20.00	19.36300	18.58920	18.8430	....	....	....	21.00	8.1287	......	5.0076

TABLE VI.—LENGTH OF CONTACT BETWEEN MATING $14\frac{1}{2}$-DEG. GENERATED GEARS IN TERMS OF TOOTH INTERVALS

Number of gear teeth	Number of teeth in pinion															
	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
12	0.034															
13	0.100	0.166														
14	0.166	0.232	0.298													
15	0.231	0.297	0.363	0.428												
16	0.297	0.362	0.428	0.494	0.559											
17	0.362	0.427	0.494	0.559	0.624	0.689										
18	0.427	0.493	0.559	0.624	0.689	0.754	0.819									
19	0.492	0.558	0.624	0.689	0.754	0.819	0.884	0.950								
20	0.557	0.623	0.689	0.754	0.819	0.884	0.949	1.014	1.080							
21	0.622	0.688	0.754	0.819	0.884	0.949	1.014	1.079	1.144	1.210						
22	0.687	0.753	0.819	0.884	0.949	1.014	1.079	1.144	1.209	1.274	1.340					
23	0.752	0.818	0.884	0.949	1.014	1.079	1.144	1.209	1.274	1.339	1.404	1.469				
24	0.807	0.883	0.949	1.014	1.079	1.144	1.209	1.274	1.339	1.404	1.469	1.534	1.599			
25	0.807	0.889	0.970	1.049	1.128	1.205	1.274	1.339	1.404	1.469	1.534	1.599	1.664	1.729		
26	0.807	0.889	0.970	1.049	1.128	1.205	1.282	1.358	1.433	1.508	1.582	1.656	1.729	1.794	1.859	
27	0.807	0.889	0.970	1.049	1.128	1.205	1.282	1.358	1.433	1.508	1.582	1.656	1.729	1.892	1.875	1.906
28	0.807	0.889	0.970	1.049	1.128	1.205	1.282	1.358	1.433	1.508	1.582	1.656	1.729	1.802	1.875	1.913
29	0.807	0.889	0.970	1.049	1.128	1.205	1.282	1.358	1.433	1.508	1.582	1.656	1.729	1.802	1.875	1.919
30	0.807	0.889	0.970	1.049	1.128	1.205	1.282	1.358	1.433	1.508	1.582	1.656	1.729	1.802	1.875	1.926
31	0.807	0.889	0.970	1.049	1.128	1.205	1.282	1.358	1.433	1.508	1.582	1.656	1.729	1.802	1.875	1.932
32	0.807	0.889	0.970	1.049	1.128	1.205	1.282	1.358	1.433	1.508	1.582	1.656	1.729	1.802	1.875	1.938
33	0.807	0.889	0.970	1.049	1.128	1.205	1.282	1.358	1.433	1.508	1.582	1.656	1.729	1.802	1.875	1.944
34	0.807	0.889	0.970	1.049	1.128	1.205	1.282	1.358	1.433	1.508	1.582	1.656	1.729	1.802	1.875	1.947
35	0.807	0.889	0.970	1.049	1.128	1.205	1.282	1.358	1.433	1.508	1.582	1.656	1.729	1.802	1.875	1.947
36	0.807	0.889	0.970	1.049	1.128	1.205	1.282	1.358	1.433	1.508	1.582	1.656	1.729	1.802	1.875	1.947
37	0.807	0.889	0.970	1.049	1.128	1.205	1.282	1.358	1.433	1.508	1.582	1.656	1.729	1.802	1.875	1.947
38	0.807	0.889	0.970	1.049	1.128	1.205	1.282	1.358	1.433	1.508	1.582	1.656	1.729	1.802	1.875	1.947
39	0.807	0.889	0.970	1.049	1.128	1.205	1.282	1.358	1.433	1.508	1.582	1.656	1.729	1.802	1.875	1.947
40	0.807	0.889	0.970	1.049	1.128	1.205	1.282	1.358	1.433	1.508	1.582	1.656	1.729	1.802	1.875	1.947
Rack........	0.807	0.889	0.970	1.049	1.128	1.205	1.282	1.358	1.433	1.508	1.582	1.656	1.729	1.802	1.875	1.947

tooth intervals or more. A 20-tooth pinion with a 25-tooth gear or larger, and a 21-tooth pinion with a 24-tooth gear or larger, and a 22-tooth pinion with a 23-tooth gear or larger will also give this contact. Smaller gears than these give so little contact that their use is questionable.

The amount of contact is but one element of gear-tooth design. Sometimes a tooth form with less contact is more favorable for the smooth transmission of power than one with more contact. This is true when the longer contact is obtained by using that part of the involute profile at or very near the base circle. Here the form is very sensitive and very difficult to make accurately because of the small and rapidly changing radius of curvature of the profile.

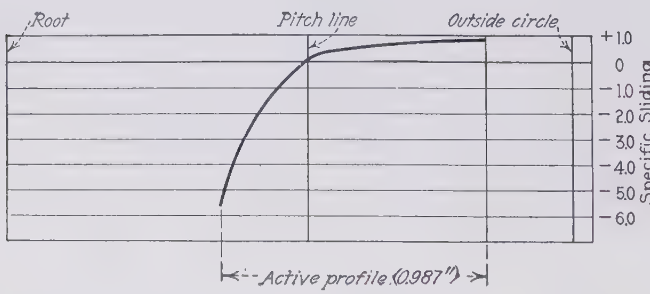


FIG. 49.—Specific sliding between two 22-tooth, 14½-deg. generated tooth gears

The information given in the first table enables an analysis of these features to be made readily. For example, we will examine the pair of 22-tooth gears. The minimum radius of curvature of the active profile in this case is at the end of the undercut and is equal to $\sqrt{Q^2 - a^2}$, which equals 0.7163 in. In many respects, the tooth action here is better with the undercut than it would be without it. The additional contact that would be secured by the elimination of this undercut would be of little value, and the more sensitive profile would be a liability rather than an asset.

The sliding conditions for this pair of gears can also be readily established. We have, from the second chapter,

$$\text{Specific sliding on pinion} = \frac{b_1N_2 - b_2N_1}{b_1N_2}$$

$$\text{Specific sliding on gear} = \frac{b_2N_1 - b_1N_2}{b_2N_1}$$

where b_1 = radius of curvature of any point of tooth profile on pinion
b_2 = radius of curvature of mating point of tooth profile on gear
N_1 = number of teeth in pinion
N_2 = number of teeth in gear

We know that at the pitch line the specific sliding is zero. We will determine the specific sliding at two other points, the beginning and the ending of contact.

At the beginning of contact, b_1 is equal to the minimum radius of curvature of the active profile, or 0.7163, in., while b_2 is equal to the total length of the line of action minus b_1. In this case, the total length of the line of action is equal to $2R \sin \alpha$, which equals 5.5084 in.; whence, b_2 equals 4.7921 in. N_2 and N_1 are both equal to 22 and cancel from the equation. Thus, we have

$$\text{Specific sliding at beginning of contact} = \frac{0.7163 - 4.7921}{0.7163} = -5.69$$

The values of b_1 and b_2 are reversed at the ending of contact, whence, we have

$$\text{Specific sliding at ending of contact} = \frac{4.7921 - 0.7163}{4.7921} = +0.85$$

This specific sliding is plotted in Fig. 49.

We will also determine the specific-sliding conditions on this same 22-tooth pinion when meshing with a 40-tooth gear. The minimum radius of curvature of the active profile of the pinion is the same as before. The total length of the line of action is equal to the sum of $R \sin \alpha$ for one gear plus $R \sin \alpha$ for the other, which is equal to 7.7618 in. Whence, we have, for the sliding conditions at the beginning of contact,

$$\begin{aligned} b_1 &= 0.7163 \text{ in.} \\ N_1 &= 22 \text{ teeth} \\ b_2 &= 7.0455 \text{ in.} \\ N_2 &= 40 \text{ teeth.} \end{aligned}$$

$$\text{Specific sliding on pinion} = \frac{0.7163 \times 40 - 7.0455 \times 22}{0.7163 \times 40} = -4.40$$

$$\text{Specific sliding on gear} = \frac{7.0455 \times 22 - 0.7163 \times 40}{7.0455 \times 22} = +0.81$$

In this case, the active profile of the pinion extends to the tip of the tooth, so that at the ending of contact the radius of curvature on the pinion is equal to $\sqrt{E^2 - a^2}$, which equals 5.5304

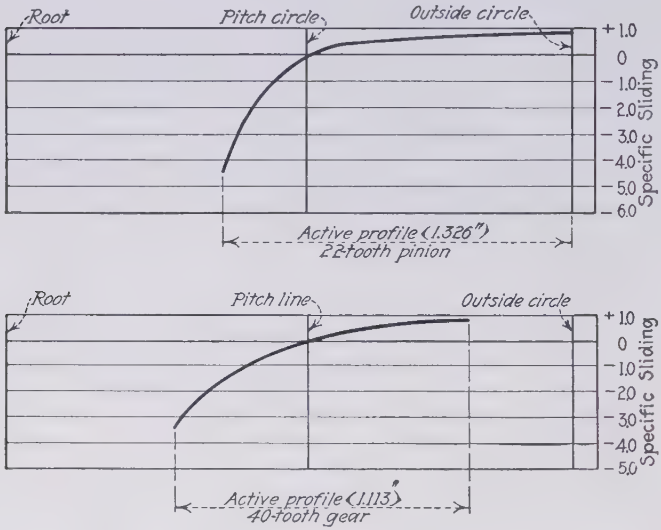


FIG. 50.—Specific sliding between a 22-tooth pinion and a 40-tooth gear, 14½-deg. generated tooth system.

in. Thus, we have, for the sliding conditions at the ending of contact,

$$\text{Specific sliding on pinion} = \frac{5.5304 \times 40 - 2.2314 \times 22}{5.5304 \times 40} = +0.77$$

$$\text{Specific sliding on gear} = \frac{2.2314 \times 22 - 5.5304 \times 40}{2.2314 \times 22} = -3.50$$

This specific sliding is plotted in Fig. 50.

THE 20-DEG., FULL-DEPTH, TOOTH-GEAR SYSTEM

The need of effective gears of smaller numbers of teeth than could be made satisfactorily with the 14½-deg. generated system led to the introduction of other gear-tooth systems. Among them was the 20-deg. full-depth tooth system. The tooth proportions of this system are the same as for milled gears, but the

pressure angle is increased to 20 deg. which results in smaller base circles and permits gears to be made with fewer teeth without excessive undercutting. In addition, due to the wider angle, the tooth forms are stronger than those of the $14\frac{1}{2}$-deg. system. The form of the basic rack of this system is shown in Fig. 51

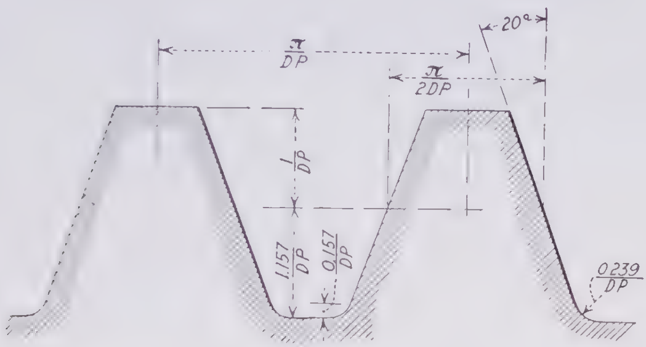


FIG. 51.—Basic rack of the 20-deg., full-depth tooth-gear system

Analysis of 20-deg. Full-depth Tooth System.—A similar analysis to that of the $14\frac{1}{2}$-deg. generated gear system will be made of this 20-deg. full-depth tooth system. We will, as before, first examine the 12-tooth pinion of this system.

Let E = outside radius
R = pitch radius
a = radius of base circle
F = addendum of basic rack, including clearance fillet
f = clearance on generating rack or hob
A = minimum root radius without undercut
H = root radius
α = pressure angle
e = excess depth of root beyond undercut limit
x = radial height above base circle of involute profile removed
Q = radius to top of undercut
N = number of teeth
Pn = normal pitch

We have the following values for a 12-tooth pinion of 1 d.p.

$$\begin{aligned} E &= 7.0000 \text{ in.} \\ R &= 6.0000 \text{ in.} \\ H &= 4.8430 \text{ in.} \\ F &= 1.1570 \text{ in.} \\ f &= 0.1570 \text{ in.} \\ \alpha &= 20 \text{ deg.} \end{aligned}$$

$a = \mathrm{R} \cos \alpha = 5.6381$ in. (see Eq. (55))
$A = R \cos^2 \alpha - f = 5.1411$ in. (see Eq. (60))
$e = A - H = 0.2981$ in.

$$x = \frac{1.5158e^2}{R} = 0.0225 \text{ in. (see Eq. (65))}$$

$$Q = a + x = 5.6606 \text{ in.}$$

The shape of this pinion tooth is shown in Fig. 52.

We will next examine the contact conditions between this pinion and a rack. We have seen before that the contact between a rack and an undercut gear, in terms of tooth intervals, is equal to

$$\frac{\sqrt{E^2 - a^2} - \sqrt{Q^2 - a^2}}{Pn}$$

In this example,

$$\sqrt{E^2 - a^2} = 4.1486$$

$$\sqrt{Q^2 - a^2} = 0.5035$$

$$Pn = \frac{2\pi a}{N} = 2.9521 \text{ in.}$$

Whence, contact with rack = 1.234 tooth intervals.

We have also the following equation for determining the contact between two similar undercut gears, when the undercut extends into the active profiles:

$$\text{Contact} = \frac{2(R \sin \alpha - \sqrt{Q^2 - a^2})}{Pn}$$

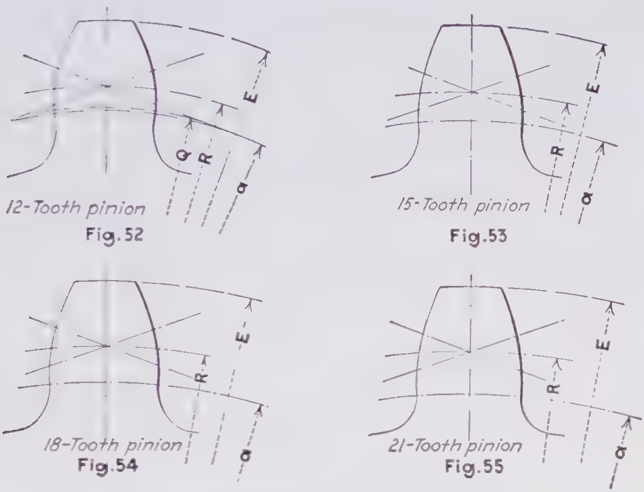
In this example, $R \sin \alpha = 2.0521$ in.
Whence, contact = 1.049 tooth intervals

The contact in both cases is sufficient to give an overlap but hardly sufficient for the quiet transmission of power, for which purpose a contact of 1.40 tooth intervals is desirable, if possible.

The accompanying Table VII gives values of the amount of undercut and contact for 1-d.p. gears of small numbers of teeth, made to the 20-deg., full-depth tooth system.

The forms of 12-, 15-, 18-, and 21-tooth gears made to this system are shown in Figs. 52, 53, 54, and 55.

A second table is given that shows the amount of contact that exists between gears of small tooth numbers made to this 20-deg., full-depth tooth system. The values are expressed in terms of tooth intervals.



FIGS. 52, 53, 54 and 55.—Tooth proportions of 12-, 15-, 18-, and 21-tooth gears. 1 d.p., of the 20-deg., full-depth tooth system.

It will be seen from this table of contact that two 14-tooth pinions are the smallest pair that give a contact of 1.40 tooth intervals or more. It can also be readily seen that this system is more desirable for gears of small tooth numbers than the $14\frac{1}{2}$-deg. generated gear system.

As before, we will examine other features of the contact conditions. The first table gives data that enable the minimum radius of curvature of the active profile and the specific sliding to be determined readily. As a first example, we must examine a pair of 14-tooth pinions. In this case, as before, the minimum radius of curvature of the active profile is at the end of the undercut and is equal to $\sqrt{Q^2 - a^2}$, which equals 0.3054 in.

We will also determine the specific sliding conditions. We have, from Eqs. (23) and (24),

$$\text{Specific sliding on pinion} = \frac{b_1N_2 - b_2N_1}{b_1N_2}$$

$$\text{Specific sliding on gear} = \frac{b_2N_1 - b_1N_2}{b_2N_1}$$

where b_1 = radius of curvature of any point on pinion tooth profile
b_2 = radius of curvature of mating point on gear tooth profile
N_1 = number of teeth in pinion
N_2 = number of teeth in gear

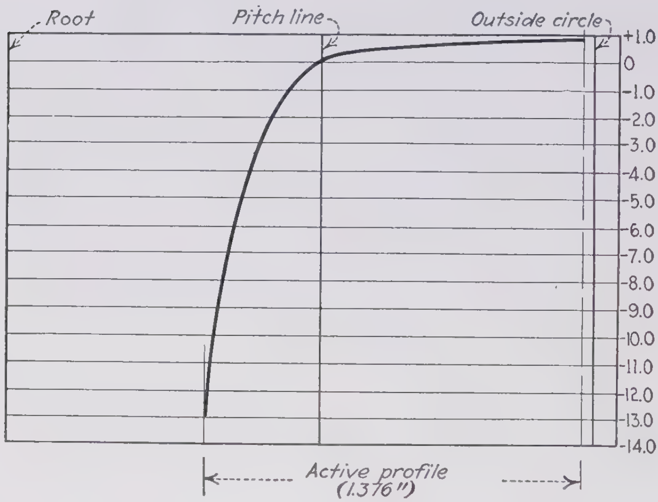


FIG. 56.—Specific sliding between two 14-tooth, 20-deg., full-depth tooth gears.

At the beginning of contact, b_1 is equal to the minimum radius of curvature of the active profile, which equals 0.3054 in.; b_2 is equal to the total length of the line of action minus b_1. In this example, the total length of the line of action is equal to $2R \sin \alpha$, which equals 4.7882 in. Whence, b_2 equals 4.4828 in.; N_1 and N_2 are both equal to 14 and cancel from the equation. Thus, we have

$$\text{Specific sliding at beginning of contact} = \frac{0.3054 - 4.4828}{0.3054} = -13.67$$

The values of b_1 and b_2 are reversed at the ending of contact, whence, we have

$$\text{Specific sliding at ending of contact} = \frac{4.4828 - 0.3054}{4.4828} = +0.93$$

This specific sliding is plotted in Fig. 56.

We will also determine the specific-sliding conditions on this same 14-tooth pinion when it meshes with a 30-tooth gear. The minimum radius of curvature of the active profile of this pinion is the same as before. The total length of the line of action is equal to the sum of the respective $R \sin \alpha$'s for each gear; this is equal to 7.5244 in. Whence, we have, for the sliding conditions at the beginning of contact,

$$b_1 = 0.3054 \text{ in.}$$
$$N_1 = 14 \text{ teeth}$$
$$b_2 = 7.2190 \text{ in.}$$
$$N_2 = 30 \text{ teeth.}$$

$$\text{Specific sliding on pinion} = \frac{0.3054 \times 30 - 7.2190 \times 14}{0.3054 \times 30} = -10.03$$

$$\text{Specific sliding on gear} = \frac{7.2190 \times 14 - 0.3054 \times 30}{7.2190 \times 14} = +0.91$$

In this example, the active profile of the pinion extends to the tip of the tooth, so that at the ending of contact, the radius of curvature on the pinion is equal to $\sqrt{E^2 - a^2}$, which equals 4.5532 in. Thus, we have, for the sliding conditions at the ending of contact,

$$b_1 = 4.5532 \text{ in.}$$
$$N_1 = 14 \text{ teeth}$$
$$b_2 = 2.9712 \text{ in.}$$
$$N_2 = 30 \text{ teeth}$$

$$\text{Specific sliding on pinion} = \frac{4.5532 \times 30 - 2.9712 \times 14}{4.5532 \times 30} = +0.69$$

$$\text{Specific sliding on gear} = \frac{2.9712 \times 14 - 4.5532 \times 30}{2.9712 \times 14} = -2.28$$

This specific sliding is plotted in Fig. 57.

A comparison of these charts with those shown in Fig. 50 shows that the contact in these smaller gears has been secured at the expense of more sensitive tooth profiles. This is inevitable

TABLE VII.—PROPORTIONS OF 20-DEG., FULL-DEPTH, TOOTH-GEAR SYSTEM

Number of teeth N	Radius of pitch circle R	Radius of base circle a	Undercut radius A	Root radius H	Excess depth of cut e	Height of undercut x	Radius to top of undercut Q	Outside radius E	$\sqrt{E^2 - a^2}$	$\sqrt{Q^2 - a^2}$	$R \sin \alpha$
12	6.00	5.63814	5.14112	4.8430	0.29812	0.02245	5.66059	7.00	4.1486	0.5035	2.0521
13	6.50	6.10799	5.58263	5.3430	0.23963	0.01339	6.12138	7.50	4.3522	0.4045	2.2231
14	7.00	6.57783	6.02414	5.8430	0.18114	0.00710	6.58493	8.00	4.5532	0.3054	2.3941
15	7.50	7.04768	6.46565	6.3430	0.12265	0.00304	7.05072	8.50	4.7518	0.2064	2.5652
16	8.00	7.51752	6.90716	6.8430	0.06416	0.00078	7.51830	9.00	4.9484	0.1074	2.7362
17	8.50	7.98737	7.34867	7.3430	0.00567	0.00001	7.98738	9.50	5.1431	0.0084	2.9072
18	9.00	8.45721	7.79018	7.8430	0.00000	0.00000	....	10.00	5.3362	......	3.0782
19	9.50	8.92706	8.23169	8.3430	....	....	....	10.50	5.5278	......	3.2492
20	10.00	9.39690	8.67320	8.8430	....	....	....	11.00	5.7182	......	3.4202
21	10.50	9.86675	9.11471	9.3430	....	....	....	11.50	5.9073	......	3.5912
22	11.00	10.33659	9.55622	9.8430	....	....	....	12.00	6.0954	......	3.7622
23	11.50	10.80644	9.99773	10.3430	....	....	....	12.50	6.2825	......	3.9332
24	12.00	11.27628	10.43924	10.8430	....	....	....	13.00	6.4687	......	4.1042
25	12.50	11.74613	10.88075	11.3430	....	....	....	13.50	6.6541	......	4.2753
26	13.00	12.21597	11.32226	11.8430	....	....	....	14.00	6.8388	......	4.4463
27	13.50	12.68582	11.76377	12.3430	....	....	....	14.50	7.0227	......	4.6173
28	14.00	13.15566	12.20528	12.8430	....	....	....	15.00	7.2061	......	4.7883
29	14.50	13.62551	12.64679	13.3430	....	....	....	15.50	7.3888	......	4.9593
30	15.00	14.09535	13.08830	13.8430	....	....	....	16.00	7.5710	......	5.1303

TABLE VIII.—LENGTH OF CONTACT BETWEEN MATING 20-DEG., FULL-DEPTH TOOTH GEARS IN TERMS OF TOOTH INTERVALS

Number of gear teeth	Number of teeth in pinion															
	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
12	1.049															
13	1.140	1.232														
14	1.232	1.323	1.415													
15	1.234	1.337	1.438	1.481												
16	1.234	1.337	1.438	1.490	1.498											
17	1.234	1.337	1.438	1.497	1.506	1.514										
18	1.234	1.337	1.438	1.505	1.514	1.522	1.529									
19	1.234	1.337	1.438	1.512	1.521	1.529	1.536	1.543								
20	1.234	1.337	1.438	1.519	1.527	1.535	1.542	1.549	1.556							
21	1.234	1.337	1.438	1.525	1.533	1.541	1.548	1.555	1.562	1.569						
22	1.234	1.337	1.438	1.531	1.539	1.547	1.554	1.561	1.568	1.574	1.580					
23	1.234	1.337	1.438	1.536	1.545	1.553	1.560	1.567	1.574	1.580	1.586	1.591				
24	1.234	1.337	1.438	1.539	1.550	1.558	1.565	1.572	1.579	1.585	1.591	1.596	1.601			
25	1.234	1.337	1.438	1.539	1.555	1.563	1.570	1.577	1.584	1.590	1.596	1.601	1.606	1.611		
26	1.234	1.337	1.438	1.539	1.559	1.567	1.574	1.581	1.588	1.594	1.600	1.605	1.610	1.615	1.620	
27	1.234	1.337	1.438	1.539	1.564	1.572	1.579	1.586	1.593	1.599	1.605	1.610	1.615	1.620	1.625	1.629
28	1.234	1.337	1.438	1.539	1.568	1.576	1.583	1.590	1.597	1.603	1.609	1.614	1.619	1.624	1.629	1.633
29	1.234	1.337	1.438	1.539	1.572	1.580	1.587	1.594	1.601	1.607	1.613	1.618	1.623	1.628	1.633	1.637
30	1.234	1.337	1.438	1.539	1.576	1.584	1.591	1.598	1.605	1.611	1.617	1.622	1.627	1.632	1.637	1.641
Rack........	1.234	1.337	1.438	1.539	1.639	1.739	1.755	1.762	1.768	1.774	1.780	1.786	1.791	1.796	1.800	1.805

with small tooth numbers. In order to secure greater contact, more sensitive, and troublesome, tooth profiles must be employed. It is possible, however, to minimize these troublesome conditions by suitable tooth design. This subject will be considered further in a succeeding chapter.

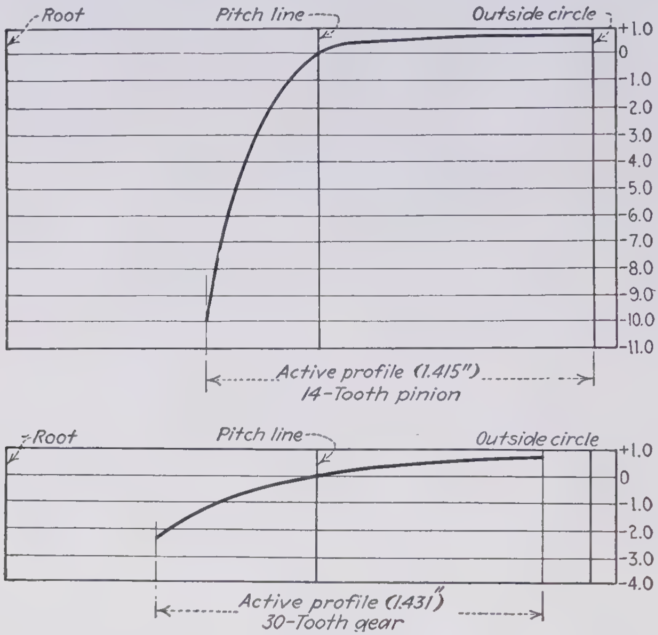


FIG. 57.—Specific sliding between a 14-tooth pinion and a 30-tooth gear, 20-deg., full-depth tooth gears.

In order to make a direct comparison with 14½-deg. generated gears, we will examine the sliding conditions on a pair of 22-tooth, 20-deg., full-depth tooth gears. Here, the contact extends to the tip of the tooth, so that the maximum radius of curvature of the active profile is equal to $\sqrt{E^2 - a^2}$, which equals 6.0954 in. The minimum radius of curvature is equal to the total length of the line of action minus the maximum radius of curvature. The total length of the line of action is equal to $2R \sin \alpha$, which equals 7.5244 in., whence the minimum radius of curvature of the active profile is equal to 1.4290 in. Thus, we have the sliding conditions at the beginning of contact

$$\text{Specific sliding} = \frac{1.4290 - 6.0954}{1.4290} = -3.26$$

The values of b_1 and b_2 are reversed at the end of contact, whence, we have

$$\text{Specific sliding} = \frac{6.0954 - 1.4290}{6.0954} = +0.76$$

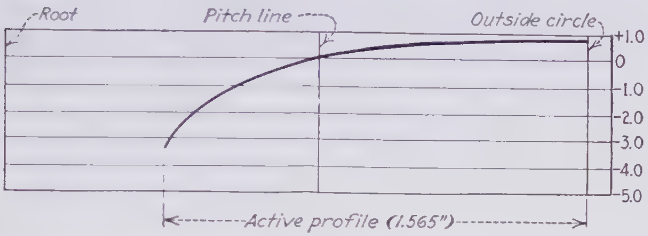


FIG. 58.—Specific sliding between two 22-tooth, 20-deg., full-depth tooth gears.

This specific sliding is plotted in Fig. 58. A comparison of this chart with that shown in Fig. 49 shows more favorable conditions for the 20-deg., full-depth tooth.

THE 20-DEG. STUB-TOOTH GEAR SYSTEM

Another gear-tooth system, introduced to meet the need of effective gears of small tooth numbers, is the 20-deg. stub-tooth system. This system not only increased the pressure angle from the conventional $14\frac{1}{2}$ deg. but also reduced the tooth height,

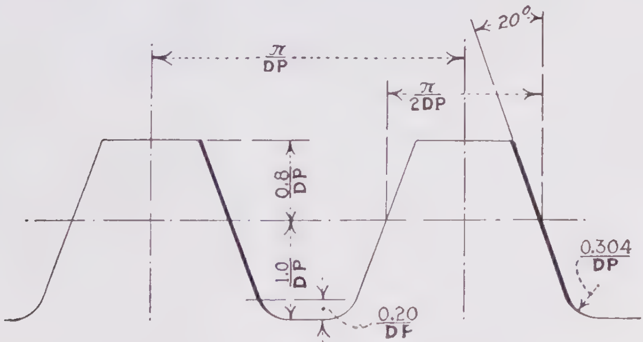


FIG. 59.—Basic rack of the 20-deg., stub-tooth gear system.

thus employing different tooth proportions. Several different stub-tooth systems have been widely used, most of them with a pressure angle of 20 deg. but with slightly different tooth heights.

The following 20-deg. stub-tooth gear system has been adopted by the American Gear Manufacturers' Association and also by a sectional committee of the American Engineering Standards Committee as a tentative standard. This form will interchange with all the existing 20-deg. stub-tooth systems; the differences in tooth heights affect only the clearance. This tooth form has the following proportions:

When $d.p.$ = diametral pitch
M = module

$$\text{addendum} = \frac{0.8}{d.p.} = 0.8M$$

$$\text{dedendum} = \frac{1.0}{d.p.} = 1.0M$$

$$\text{working depth} = \frac{1.6}{d.p.} = 1.6M$$

$$\text{whole depth} = \frac{1.8}{d.p.} = 1.8M$$

$$\text{clearance} = \frac{0.2}{d.p.} = 0.2M$$

The form of the basic rack of this system is shown in Fig. 59.

When $d.p.$ = diametral pitch
$c.p.$ = circular pitch
N = number of teeth in gear

$$\text{Outside diameter} = \frac{N + 1.6}{d.p.}$$

$$\text{Pitch diameter} = \frac{d.p.}{N}$$

$$\text{Root diameter} = \frac{N - 2.0}{d.p.}$$

$$\text{Circular pitch} = \frac{3.1416}{d.p.}$$

$$\text{Thickness of tooth on pitch line} = \frac{c.p.}{2} = \frac{1.5708}{d.p.}$$

Analysis of 20-deg. Stub-tooth System.—An analysis similar to those of the two preceding gear systems will be made of this 20-deg. stub-tooth system. For the first example, we will take the 12-tooth pinion of this system.

Let E = outside radius
R = pitch radius
a = radius of base circle
F = addendum of basic rack, including clearance fillet
f = clearance on generating rack or hob
A = minimum root radius without undercut
H = root radius
α = pressure angle
e = excess depth of root beyond undercut limit
x = radial height above base circle of undercut part of profile
Q = radius of top of undercut
N = number of teeth
Pn = normal pitch

We have the following values for a 12-tooth pinion of 1 d.p.

$$E = 6.8000 \text{ in.}$$
$$R = 6.0000 \text{ in.}$$
$$H = 5.0000 \text{ in.}$$
$$F = 1.0000 \text{ in.}$$
$$f = 0.2000 \text{ in.}$$
$$\alpha = 20 \text{ deg.}$$
$$a = R \cos \alpha = 5.6381 \text{ in.} \qquad \text{(see Eq. (55))}$$
$$A = R \cos^2 \alpha - f = 5.0981 \text{ in.} \qquad \text{(see Eq. (60))}$$
$$e = A - H = 0.0981 \text{ in.}$$
$$x = \frac{1.5158e^2}{R} = 0.0024 \text{ in.} \qquad \text{(see Eq. (65))}$$
$$Q = a + x = 5.6406 \text{ in.}$$

The shape of this pinion tooth is shown in Fig. 60.

Let us examine the contact conditions between this pinion and a rack. We have seen before that the contact between a rack and an undercut pinion, in terms of tooth intervals, is equal to

$$\frac{\sqrt{E^2 - a^2} - \sqrt{Q^2 - a^2}}{Pn}$$

In this example,

$$\sqrt{E^2 - a^2} = 3.8014 \text{ in.}$$
$$\sqrt{Q^2 - a^2} = 0.1682 \text{ in.}$$
$$Pn = 2.9521 \text{ in.}$$

Whence,

$$\text{contact with rack} = 1.24 \text{ tooth intervals}$$

We will next determine the contact conditions when two 12-tooth pinions mesh together. In this example, the active profiles do not extend to the undercut. Thus, the duration of contact when two of these pinions mesh together is equal to

$$\frac{2(\sqrt{E^2 - a^2} - R \sin \alpha)}{Pn}$$

In this example,

$$R \sin \alpha = 2.0521 \text{ in.}$$

Whence, length of contact between two 12-tooth pinions is 1.19 tooth intervals.

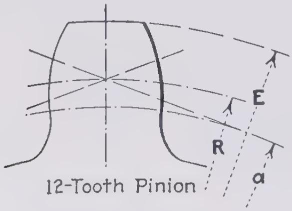


Fig. 60.

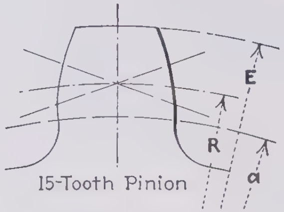


Fig. 61.

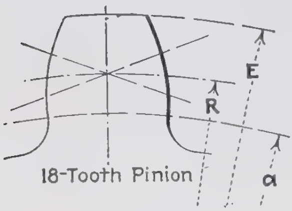


Fig. 62.

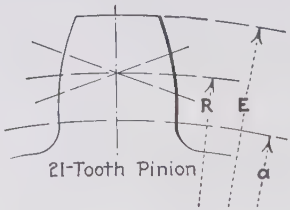


Fig. 63.

Figs. 60, 61, 62 and 63.—Tooth proportions of 12-, 15-, 18- and 21-tooth gears, 1 d.p., of the 20-deg. stub-tooth system.

The contact in both cases is sufficient to give an overlap, but its value does not amount to 1.40 intervals, which is a desirable minimum contact to secure, if possible. The contact is greater, however, than is obtained with a 12-tooth pinion made to conform with either of the two preceding gear systems, the 14½-deg. generated or the 20-deg. full-depth.

The accompanying table gives values of the amount of undercut and contact for 1-d.p. gears of small numbers of teeth of the 20-deg. stub-tooth system. With this system, only the 12- and

13-tooth pinions are undercut, and the amount of undercut is so small that it has no appreciable effect on their action.

The forms of 12-, 15-, 18-, and 21-tooth gears made to this 20-deg. stub-tooth system are shown in Figs. 60, 61, 62, and 63, respectively.

Table X shows the amount of contact that exists between mating gears of small tooth numbers in this system. The values are expressed in terms of tooth intervals.

It will be seen from this table of contact that none of the combinations of gears with small tooth numbers gives a contact of 1.40 tooth intervals. As stated before, the final design of any gear-tooth form is a compromise between several conflicting elements. Here, the amount of contact has been decreased, in order to avoid excessive undercutting. This shortened contact imposes the necessity of greater care in production, if quiet running gears are to be obtained.

As before, we will examine other features of the contact conditions. The first table gives data that enable the minimum radius of curvature of the active profile and the specific sliding to be determined readily.

As a first example, we will examine a pair of 12-tooth pinions of 1 d.p. The active profile extends to the tip of the tooth, so that the maximum radius of curvature is equal to $\sqrt{E^2 - a^2}$, which equals 3.8014 in. The minimum radius of curvature of the active profile is equal to the total length of the line of action minus the maximum radius of curvature. The total length of the line of action is equal to $2R \sin \alpha$, which equals 4.1042 in., whence, the minimum radius of curvature of the active profile is equal to 0.3028 in.

We will also determine the specific sliding conditions. We have, from a previous chapter,

$$\text{Specific sliding on pinion} = \frac{b_1N_2 - b_2N_1}{b_1N_2}$$

$$\text{Specific sliding on gear} = \frac{b_2N_1 - b_1N_2}{b_2N_1}$$

Where b_1 = radius of curvature of any point on pinion-tooth profile

b_2 = radius of curvature of mating point on gear tooth

N_1 = number of teeth in pinion

N_2 = number of teeth in gear

In this example, we have, at the beginning of contact,

$$b_1 = 0.3028 \text{ in.}$$
$$b_2 = 3.8014 \text{ in.}$$
$$N_1 \text{ and } N_2 = 12,$$

whence,

Specific sliding at beginning of contact =

$$\frac{0.3028 - 3.8014}{0.3028} = -11.55$$

The values of b_1 and b_2 are reversed at the ending of contact, whence, we have

$$\text{Specific sliding at ending of contact} = \frac{3.8014 - 0.3028}{3.8014} = +0.92$$

The specific sliding is plotted in Fig. 64.

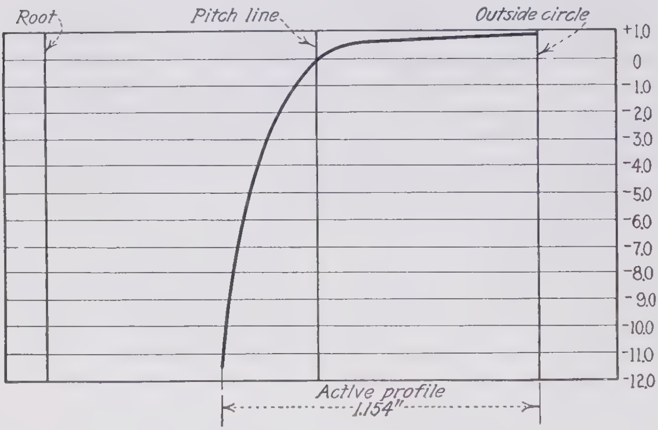


FIG. 64.—Specific sliding between two 12-tooth, 20-deg., stub-tooth gears.

We will also determine the specific-sliding conditions on this same 12-tooth pinion in mesh with a 30-tooth gear. The maximum radius of curvature of the active profile of the pinion is the same as before, but, as contact takes place down to the top of the undercut, the minimum radius of curvature of the active profile becomes equal to $\sqrt{Q^2 - a^2}$, which is equal to 0.1682 in. The corresponding radii of curvature on the 30-tooth gear are obtained by subtracting these pinion radii from the total length of the line of action, which equals 7.1824 in. Whence, we have, for the sliding conditions at the beginning of contact,

$$b_1 = 0.1682 \text{ in.}$$
$$N_1 = 12 \text{ teeth}$$
$$b_2 = 7.0142 \text{ in.}$$
$$N_2 = 30 \text{ teeth.}$$

Specific sliding on pinion =

$$\frac{0.1682 \times 30 - 7.0142 \times 12}{0.1682 \times 30} = -15.68$$

$$\text{Specific sliding on gear} = \frac{7.0142 \times 12 - 0.1681 \times 30}{7.0142 \times 12} = +0.94$$

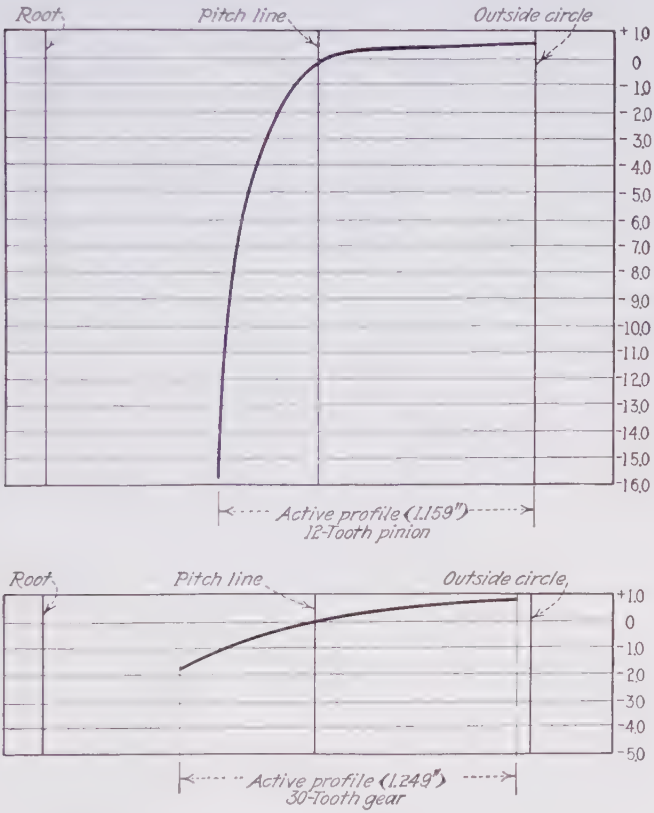


FIG. 65.—Specific sliding between a 12-tooth pinion and a 30-tooth gear, 20-deg. stub-tooth system.

And we have, for the sliding conditions at the ending of contact,

$$b_1 = 3.8014 \text{ in.}$$
$$b_2 = 3.3810 \text{ in.}$$

Specific sliding on pinion =

$$\frac{3.8014 \times 30 - 3.3810 \times 12}{3.8014 \times 30} = +0.64$$

$$\text{Specific sliding on gear} = \frac{3.3810 \times 12 - 3.8014 \times 30}{3.3810 \times 12} = -1.81$$

This specific sliding is plotted in Fig. 65.

As a direct comparison with gears of the other two systems, we will examine the sliding conditions on a pair of 22-tooth gears, 1 d.p., made to the 20-deg. stub-tooth system. Here the contact extends to the top of the tooth, so that the maximum radius of curvature of the active profile is equal to $\sqrt{E^2 - a^2}$, which equals 5.6916 in. The minimum radius of curvature is obtained by subtracting this maximum radius from the total length of the

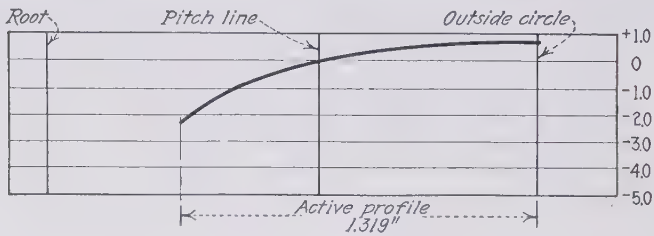


FIG. 66.—Specific sliding between two 22-tooth, 20-deg., stub-tooth gears.

line of action, which gives us 1.7328 in. as the minimum radius of curvature of the active profile. Thus, we have at the beginning of contact

$$b_1 = 1.7328 \text{ in.}$$
$$b_2 = 5.6916 \text{ in.}$$
$$N_1 \text{ and } N_2 = 22 \text{ teeth}$$

Specific sliding at beginning of contact =

$$\frac{1.7328 - 5.6916}{1.7328} = -2.28$$

The values of b_1 and b_2 are reversed at the ending of contact, whence, we have

$$\text{Specific sliding at ending of contact} = \frac{5.6916 - 1.7328}{5.6916} = +0.69$$

This specific sliding is plotted in Fig. 66. The comparison of this chart with those shown in Figs. 49 and 58 show most favorable conditions for the 20-deg. stub tooth.

TABLE IX.—PROPORTIONS OF 20-DEG. STUB-TOOTH GEAR SYSTEM

Number of teeth N	Radius of pitch circle R	Radius of base circle a	Undercut radius A	Root radius H	Excess depth of cut e	Height of undercut x	Radius to top of undercut Q	Outside radius E	$\sqrt{E^2 - a^2}$	$\sqrt{Q^2 - a^2}$	$R \sin \alpha$
12	6.00	5.63814	5.09812	5.000	0.09812	0.00243	5.64057	6.80	3.8014	0.1682	2.0521
13	6.50	6.10799	5.53963	5.500	0.03963	0.00037	6.10836	7.30	3.9978	0.0668	2.2231
14	7.00	6.57783	5.98114	6.000	0.00000	0.00000	....	7.80	4.1919	......	2.3941
15	7.50	7.04768	6.42265	6.500			....	8.30	4.3840	......	2.5652
16	8.00	7.51752	6.86416	7.000	....	....	....	8.80	4.5745	......	2.7362
17	8.50	7.98737	7.30567	7.500	....	....	....	9.30	4.7635	......	2.9072
18	9.00	8.45721	7.74718	8.000	....	....	....	9.80	4.9513	......	3.0782
19	9.50	8.92706	8.18869	8.500	....	....	....	10.30	5.1378	......	3.2492
20	10.00	9.39690	8.63020	9.000	....	....	....	10.80	5.3233	......	3.4202
21	10.50	9.86675	9.07171	9.500	....	....	....	11.30	5.5079	......	3.5912
22	11.00	10.33659	9.51322	10.000	....	....	....	11.80	5.6916	......	3.7622
23	11.50	10.80644	9.95473	10.500	....	....	....	12.30	5.8745	......	3.9332
24	12.00	11.27628	10.39624	11.000	....	....	....	12.80	6.0568	......	4.1042
25	12.50	11.74613	10.83775	11.500	....	....	....	13.30	6.2384	......	4.2753
26	13.00	12.21597	11.27926	12.000	....	....	....	13.80	6.4194	......	4.4463
27	13.50	12.68582	11.72077	12.500	....	....	....	14.30	6.6000	......	4.6173
28	14.00	13.15566	12.16228	13.000	....	....	....	14.80	6.7800	......	4.7883
29	14.50	13.62551	12.60379	13.500	....	....	....	15.30	6.9595	......	4.9593
30	15.00	14.09535	13.04530	14.000	....	....	....	15.80	7.1386	......	5.1303

TABLE X.—LENGTH OF CONTACT BETWEEN MATING, 20-DEG., STUB-TOOTH GEARS IN TERMS OF TOOTH INTERVALS

Number of teeth in gear	Number of teeth in pinion															
	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
12	1.185															
13	1.193	1.202														
14	1.201	1.210	1.217													
15	1.208	1.217	1.225	1.232												
16	1.215	1.224	1.232	1.239	1.245											
17	1.221	1.230	1.238	1.245	1.251	1.257										
18	1.227	1.235	1.243	1.250	1.256	1.262	1.268									
19	1.237	1.240	1.248	1.255	1.261	1.267	1.273	1.279								
20	1.237	1.245	1.253	1.260	1.266	1.272	1.278	1.284	1.289							
21	1.237	1.250	1.258	1.265	1.271	1.277	1.283	1.289	1.294	1.298						
22	1.237	1.254	1.262	1.269	1.275	1.281	1.287	1.293	1.298	1.302	1.307					
23	1.237	1.258	1.266	1.273	1.279	1.285	1.291	1.297	1.302	1.306	1.311	1.315				
24	1.237	1.262	1.270	1.277	1.283	1.289	1.295	1.301	1.306	1.310	1.316	1.320	1.323			
25	1.237	1.266	1.274	1.281	1.287	1.293	1.299	1.305	1.310	1.314	1.320	1.324	1.327	1.330		
26	1.237	1.269	1.277	1.284	1.290	1.296	1.302	1.308	1.313	1.317	1.322	1.326	1.329	1.333	1.336	
27	1.237	1.272	1.280	1.287	1.293	1.299	1.305	1.311	1.316	1.320	1.325	1.329	1.332	1.336	1.340	1.343
28	1.237	1.275	1.283	1.290	1.296	1.302	1.308	1.314	1.319	1.323	1.328	1.332	1.335	1.339	1.343	1.346
29	1.237	1.278	1.286	1.293	1.299	1.305	1.311	1.317	1.322	1.326	1.331	1.335	1.338	1.342	1.346	1.349
30	1.237	1.281	1.289	1.296	1.302	1.308	1.314	1.320	1.325	1.329	1.334	1.338	1.341	1.345	1.348	1.351
Rack........	1.237	1.331	1.401	1.408	1.415	1.421	1.426	1.432	1.436	1.441	1.445	1.449	1.453	1.457	1.460	1.463

This 20-deg. stub-tooth form is very widely used for automobile transmission gears where the tooth numbers are small. It is also used to some extent for heavy mill gears with large tooth numbers, where the increased strength of the shorter teeth is a distinct asset.

In general, however, the $14\frac{1}{2}$-deg. generated gear system is best adapted for gears with large tooth numbers, from 40 to 50 teeth and larger, while the 20-deg. systems are the best adapted to gears of small tooth numbers.

CHAPTER V

POSSIBILITIES OF INVOLUTE GEAR-TOOTH DESIGN

With but few exceptions, gear-tooth design today still follows the conventions established in the days of cast gear teeth. With all gear-tooth forms, except the involute, there is a definite pitch line from which the conjugate forms are developed, and these forms are not conjugate to any other pitch line. With the involute, the form is developed from a base circle, and any diameter on the profile is suitable for use as a pitch circle. In fact, a definite pitch line does not exist on the involute form until two involutes are placed in contact with each other. Then the position of the pitch line depends entirely upon the size of the base circles of the two involutes and the distance between their centers.

The many valuable properties of the involute form make possible a great flexibility in its use. Every restriction imposed upon the gear-tooth design, however, tends to nullify many of the possible benefits. Certain restrictions are necessary in order to obtain economy of production, but many of the restrictions now imposed are entirely unnecessary.

For example, most tooth forms employed today have equal addenda on mating gears. In order to maintain this condition, the tooth heights and pressure angles employed in any system must be constant and are usually based on those required for the smallest effective gear of the system.

Again, most gear-tooth systems are based on the use of center distances, which are proportional to the numbers of teeth in the gears. This condition automatically requires a fixed pressure angle for the entire system, although a higher pressure angle would be more effective for the smaller tooth numbers and a lower pressure angle would be more effective for the larger tooth numbers.

If gear sets were always standardized commercial parts, such as many bolts and nuts, these restrictions would be necessary for the sake of economy. As a matter of fact, there is a considerable field for the use of such standardized gears. For this use, the 14½-deg. composite form is probably the most satisfactory.

But this field, though large, covers a relatively small portion of the total gear requirements of today.

Manufacturing conditions of today are entirely changed from what they were at the time when the $14\frac{1}{2}$-deg. composite form was developed. Considerations that once were of paramount importance have little weight now. Then, the component part of a mechanism which was made in sufficient quantities to justify the expense of special tools, such as form cutters, was a great exception. Today, such component parts are common. Furthermore, methods of molding or generating the tooth forms instead of milling them with form cutters are now in general use. Such generating processes offer many opportunities for producing improved involute gear-tooth forms without the expense of special cutters.

It is true that in order to secure such improved forms, interchangeability, as the term is generally understood, will be lost. All involute gears, nevertheless, that have the same normal pitch will run together. Only proportional center distances will be lost.

Great stress is often laid upon the importance of keeping the tooth forms interchangeable at standard, or proportional, center distances. A little study will show that this feature of interchangeability is of little or no importance except for economy in tools on general jobbing work, where form-milling cutters are employed, or for the convenience of the designer who desires to use only the simplest of arithmetical calculations.

A large portion of gear drives consists only of pairs, or series of pairs, of gears. Universal interchangeability has no value here. Furthermore, a large portion of gear blanks can be used only in the particular place for which they are designed. This is evident from a study of Fig. 67, which shows some representative gears used in machine-tool construction. Universal interchangeability of gear-tooth forms has no particular value here.

The requirements and the demands that gears today are expected to meet are no mean ones and have grown with the rapid development of the manufacturing industries to such an extent that only the most accurately made and carefully designed gears can meet them. This is one reason why the manufacture of gears of all kinds is concentrated more and more in those establishments that make a specialty of it and devote all their attention and experience to it.

Conventional spur gears were unable to meet all of the more exacting requirements; hence, helical and herringbone gears were introduced. Incidentally, the center distances, at which the helical gears operate, are seldom standard. Furthermore, each pair or train is complete in itself, and the tooth forms will seldom interchange with those of other pairs or trains. Yet, although

FIG. 67.—Miscellany of gears used in machine-tool construction

they are in extensive use, this loss of interchangeability has not proved to be a great handicap.

Helical and herringbone gears are extensively used when high pitch-line speeds are required. Spur gears of standard forms are seldom used when the pitch-line velocities exceed 2,400 ft. per minute. Yet spur gears of involute form that are accurately made and designed to secure favorable involute action are being

used to transmit power when running at pitch-line velocities of over 10,000 ft. per minute.

In principle, standardization is a desirable goal, because of its many economies. But if any standardization acts as a bar to progress in the art and imposes improper functional conditions, it should be abandoned or its use should be restricted to those applications the requirements of which it meets. No standard should ever be used unless it meets the peculiar requirements of functioning and service in each individual case as well as, or better than, any other construction would meet them.

The terms "functioning" and "service" are used here in their broadest sense. Service includes the furnishing to the customer of satisfactory mechanisms at reasonable cost. The requirements that gears have to meet are not always severe, particularly when their speed is low. In such cases, certain refinements are unnecessary; the best gear is the satisfactory gear that is cheapest to produce. When the speed is high, or the amount of power to be transmitted is large, or when maximum strength with minimum weight is essential, no possible refinement can be safely ignored.

The limitations of the conventional, standardized, interchangeable systems of gear teeth have long been known. Many suggestions have been made for improvement. In some instances, increased pressure angles were proposed, the constant addendum and dedendum being still retained. In other instances, a pinion with an increased addendum was proposed, while the gear was to have a correspondingly decreased addendum, the constant tooth height and constant pressure angle being still retained.

In order to exploit the involute curve to the greatest extent as a gear-tooth profile, the pressure angles and the tooth proportions must be variable. If this proposal necessitated special cutting tools for every different gear tooth, the cost of producing such gears might be prohibitive. But if a generating process is used, a few standard cutters are sufficient to meet almost every condition, hence, the cost of production should not be increased.

The $14\frac{1}{2}$-deg. Variable-center Distance System.—As an example of what can be done with the involute form as a gear-tooth profile, we will study the possibilities of a system where a $14\frac{1}{2}$-deg. basic rack of constant proportions is used to generate a series of gears, of which a 10-tooth pinion is the smallest. The

only restriction placed upon the tooth design is that all gears of the same nominal pitch must be generated with the same cutter, or hob, of basic-rack form.

We will establish our basic-rack form first. This will be of 14½ deg. with a nominal working depth of 2.0000 in. and a clearance of 0.2000 in. for a 1-d.p. rack. This basic-rack form is shown in Fig. 68. What we may call, for want of a better term, the "nominal pitch line" of this rack, is shown in the middle of the working depth. At this point, the thickness of the tooth and the space of the basic rack are both equal to 1.5708 in.

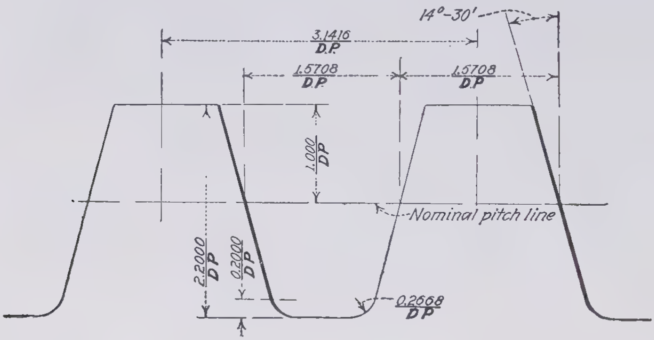


Fig. 68.—Basic rack of the 14½-deg., variable-center, involute tooth system.

We will first establish the smallest pinion of the series, that of 10 teeth. We know that the small gears of any series are the critical ones. If we cut them too deep, we should have excessive undercut. If we made them too large in diameter, we should have pointed teeth. In order to avoid both of these conditions on the 10-tooth pinion, we will make its root at the undercut limit.

When E = outside radius
R = nominal pitch radius, or 14½-deg. pitch radius
a = radius of base circle
F = addendum of basic-rack form cutter, including clearance
f = clearance on generating rack or hob
A = minimum root radius without undercut
H = root radius
α = pressure angle of basic-rack form cutter
N = number of teeth
Pn = normal pitch

We have the following values for the 10-tooth pinion of this series: In all cases, calculations will be based on a nominal diametral pitch of one.

$$N = 10 \text{ teeth}$$
$$R = 5.0000 \text{ in.}$$
$$F = 1.2000 \text{ in.}$$
$$f = 0.2000 \text{ in.}$$
$$\alpha = 14\tfrac{1}{2} \text{ deg.}$$

$$a = R \cos \alpha = 4.8407 \text{ in.} \qquad \text{(see Eq. (55))}$$
$$H = A = R \cos^2 \alpha - f = 4.4866 \text{ in.} \qquad \text{(see Eq. (60))}$$

The next step will be to determine the thickness of the tooth of this 10-tooth pinion on its 14½-deg. pitch line.

When $c.p.$ = circular pitch of basic-rack form cutter
G = distance from center of gear to top of rack tooth
T = tooth thickness of gear at R

$$T = 2 \tan \alpha (G + F - R) + \frac{c.p.}{2} \qquad \text{(see Eq. (50))}$$

In this example,

$$c.p. = 3.1416 \text{ in.}$$
$$G = H = 4.4866 \text{ in.}$$
$$F = 1.2000 \text{ in.}$$
$$R = 5.0000 \text{ in.}$$

Whence, $T = 1.9259$ in., which is the thickness of the tooth of the 10-tooth pinion on its 14½-deg. pitch line.

The next step will be to determine at what center distance two of these two 10-tooth pinions will mesh together without backlash. All of these gear calculations are made for gears without backlash. Backlash is obtained by cutting deeper. We have, from Prob. 9, in Chap. III

$$\operatorname{inv} \alpha_2 = \frac{T}{2R} - \frac{\pi}{2N} + \operatorname{inv} \alpha_1 \qquad \text{(see Eq. (45))}$$

$$C_2 = \frac{2R \cos \alpha_1}{\cos \alpha_2} \qquad \text{(see Eq. (46))}$$

Where α_1 = pressure angle of cutter
α_2 = pressure angle between gears when meshed
C_2 = center distance between gears when meshed

Whence,

$$\operatorname{inv} \alpha_2 = \frac{1.9259}{10} - \frac{3.1416}{20} + 0.005545 = 0.041056$$

From the table of involute functions in Chap. III, we get

$$\alpha_2 = 27 \text{ deg. } 36 \text{ min.}$$

Whence,

$$\cos \alpha_2 = 0.88620$$

Thus,

$$C_2 = \frac{10 \times 0.96815}{0.88620} = 10.9247 \text{ in.}$$

We will next determine the tooth depth for this pair of gears. The center distance is 10.9247 in., and since the sum of the root radii equals twice H or 8.9731 in., a distance of 1.9516 in. is left

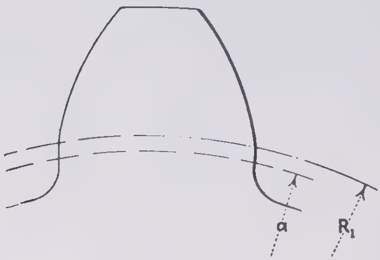


FIG. 69.—Form of 10-tooth pinion of the 14½-deg., variable-center, involute system.

between the roots of this pair of gears. This distance must include the working depth of tooth and two clearances. The clearance on the basic rack is one-tenth of the working depth, and we will maintain this proportion, whence,

$$\begin{aligned} 1.200 \times \text{working depth} &= 1.9516 \text{ in.} \\ \text{whence, working depth} &= 1.6263 \text{ in.} \\ \text{clearance} &= 0.1626 \text{ in.} \\ \text{tooth depth} &= 1.7890 \text{ in.} \\ \text{and } E = H + 1.7890 &= 6.2756 \text{ in.} \end{aligned}$$

We will now check this tooth form to be sure that the tooth is not pointed. We have, from Prob. 6, in Chap. III,

$$\cos \alpha_2 = \frac{r_1 \cos \alpha_1}{r_2} \qquad \text{(see Eq. (39))}$$

$$T_2 = 2r_2\left(\frac{T_1}{2r_1} + \operatorname{inv} \alpha_1 - \operatorname{inv} \alpha_2\right) \qquad \text{(see Eq. (40))}$$

T_2 = thickness of tooth at r_2, or outside radius in this case.
$r_2 = 6.2755$ in.
$r_1 = 5.0000$ in.
$T_1 = 1.9259$ in.
$\alpha_1 = 14\frac{1}{2}$ deg.

$$\cos \alpha_2 = \frac{4.84074}{6.2755} = 0.77137$$

$\alpha_2 = 39$ deg. 31 min. and inv $\alpha_2 = 0.13513$

$$T_2 = 12.5510\left(\frac{1.92591}{10} + 0.005545 - 0.13513\right) = 0.7907 \text{ in.}$$

The form of this pinion tooth is shown in Fig. 69.

We will also determine the radius at which this tooth becomes pointed, which will establish the maximum tooth depth that this pinion could possibly have. We have, from Prob. 7, in Chap. III,

$$\operatorname{inv}\alpha_2 = \frac{T_1}{2r_1} + \operatorname{inv}\alpha_1 \qquad \text{(see Eq. (41))}$$

$$r_2 = \frac{r_1 \cos \alpha_1}{\cos \alpha_2} \qquad \text{(see Eq. (42))}$$

Where $r_1 = 5.0000$ in. = known pitch radius
$\alpha_1 = 14\frac{1}{2}$ deg. = known pressure angle at r_1
$T_1 = 1.9259$ in. = known tooth thickness at r_1
r_2 = radius where tooth is pointed

Whence,

$$\operatorname{inv}\alpha_2 = \frac{1.92591}{10} + 0.005545 = 0.198136$$

From the table of involute functions, in Chap. III we get

$$\alpha_2 = 44° - 1'$$

Whence,

$$\cos \alpha_2 = 0.71914$$

and

$$r_2 = \frac{4.84074}{0.71914} = 6.7312 \text{ in.}$$

As the root radius equals 4.4866 in., the maximum tooth depth possible on this 10-tooth pinion would be 2.2446 in. This is greater than the basic-rack tooth depth, which is the maximum that could ever be employed.

We will now examine the exact conditions between these two pinions. We have, from Chap. III, number of teeth in contact =

$$\frac{\sqrt{E_1^2 - a_1^2} + \sqrt{E_2^2 - a_2^2} - C \sin \alpha}{Pn} \quad \text{(see Eq. (58))}$$

In this example, both gears are identical.

$$\sqrt{E^2 - a^2} = 3.9924$$

$$Pn = \frac{2\pi a}{N} = 3.0415 \text{ in.} \qquad \text{(see Eq. (57))}$$

$$\alpha = 27 \text{ deg.} - 36'$$

$$C \sin \alpha = 5.0614$$

Whence

$$\text{contact} = \frac{2.9234}{3.04153} = 0.961 \text{ tooth intervals.}$$

This contact does not give an overlap, so that this pair should not be used for the transmission of power.

We will now examine the sliding conditions on this pair. We have, from Eq. (23), in Chap. II,

$$\text{Specific sliding on pinion} = \frac{b_1 N_2 - b_2 N}{b_1 N_2}$$

where b_1 = radius of curvature of any point on pinion-tooth profile

b_2 = radius of curvature of mating point on gear-tooth profile

N_1 = number of teeth in pinion

N_2 = number of teeth in gear

In this case, the active profile of the pinion extends to the tip of the tooth, so that the maximum radius of curvature of the active profile is equal to $\sqrt{E^2 - a^2}$, which equals 3.9924 in. The minimum radius of curvature of the active profile is equal to the total length of the line of action minus the maximum radius of curvature, or 1.0690 in., in this example. Whence, we have

$$\text{Specific sliding at beginning of contact} = \frac{1.0690 - 3.9924}{1.0690} = -2.79$$

The values of b_1 and b_2 are reversed at the end of contact, whence, we have

$$\text{Specific sliding} = \frac{3.9924 - 1.0690}{3.9924} = +0.74$$

This specific sliding is plotted in Fig. 70. A comparison of this chart with Figs. 49, 56, and 64 shows that the sliding conditions on these 10-tooth pinions are much more favorable than on the 22-tooth, 14½-deg., generated gears; the 14-tooth, 20-deg., full-depth gears; and the 12-tooth, 20-deg., stub-tooth gears of the standard systems.

Although the contact on these 10-tooth pinions is too short to permit them to be used for the transmission of power, we will use them as the start of a variable-center distance system of gears. The next step will be to consider the larger gears of such

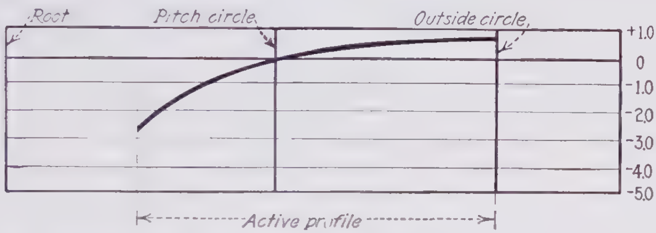


FIG. 70.—Specific sliding between two 10-tooth pinions, 14½-deg., variable-center system.

a system. We know that as the numbers of teeth become larger, smaller pressure angles become more effective. These 10-tooth pinions mesh with a pressure angle of over 27 deg. When the tooth numbers are 40 or larger, for example, the pressure angle of 14½ deg. is a very good one. We will therefore make all the gears of this system of standard proportions that have 40 teeth or more. This will mean that when the smallest gear of a pair has 40 teeth or more, the gears will mesh at proportional- or standard-center distances, with a pressure angle of 14½ deg. Thus, the need of variable-center distances will be limited to these drives where one, or more, of the gears has less than 40 teeth. Also, this variable-center distance system will be an extension of the existing 14½-deg. generated system rather than an entirely new system of gear teeth. This extension is necessary in order to obtain more effective gears of small tooth numbers.

In this system, the 40-tooth gear of 1 d.p. will therefore have the following values:

$$
\begin{aligned}
E &= 21.0000 \text{ in.} &&= \text{outside radius} \\
R &= 20.0000 \text{ in.} &&= \text{nominal pitch radius} \\
H &= 18.8000 \text{ in.} &&= \text{root radius} \\
f &= 0.2000 \text{ in.} &&= \text{clearance on generating rack} \\
\alpha &= 14\tfrac{1}{2} \text{ deg.} &&= \text{pressure angle of basic rack} \\
a &= R \cos \alpha &&= 19.3629 \text{ in.} = \text{radius of base circle} \\
T &= 1.5708 \text{ in.} &&= \text{tooth thickness at } R
\end{aligned}
$$

We will examine the contact conditions on a pair of these 40-tooth gears when they mesh together. We have, from Eq. (58), in Chap. III,

$$\text{Number of teeth in contact} = \frac{\sqrt{E_1^2 - a_1^2} + \sqrt{E_2^2 - a_2^2} - C \sin \alpha}{Pn}$$

In this example, both gears are equal; therefore,

$$
\begin{aligned}
\sqrt{E^2 - a^2} &= 8.1287 \\
Pn &= 3.0415 \text{ in.} \\
C \sin \alpha &= 10.1520 \text{ in.}
\end{aligned}
$$

whence,

$$\text{contact} = \frac{6.1054}{3.04153} = 2.007 \text{ tooth intervals}$$

This contact is ample. We will also examine the sliding conditions. We have, from Chap. II,

$$\text{Specific sliding on pinion} = \frac{b_1 N_2 - b_2 N}{b_1 N_1}$$

where b_1 = radius of curvature of any point on pinion-tooth profile
b_2 = radius of curvature of mating point on gear tooth
N_1 = number of teeth in pinion
N_2 = number of teeth in gear

In this example, the active profiles of the teeth of the gears extend to their tips, so that the maximum radius of curvature of the active profile is equal to $\sqrt{E^2 - a^2}$, which equals 8.1287 in. The minimum radius of curvature is equal to the total length of

the line of action minus this maximum radius of curvature, or 2.0233 in., in this example. Whence, we have

$$\text{Specific sliding at beginning of contact} = \frac{2.0233 - 8.1287}{2.0233} = -3.02$$

The values of b_1 and b_2 are reversed at the end of contact, whence, we have

$$\text{Specific sliding} = \frac{8.1287 - 2.0233}{8.1287} = +0.75$$

This specific sliding is plotted in Fig. 71, and it shows very favorable sliding conditions.

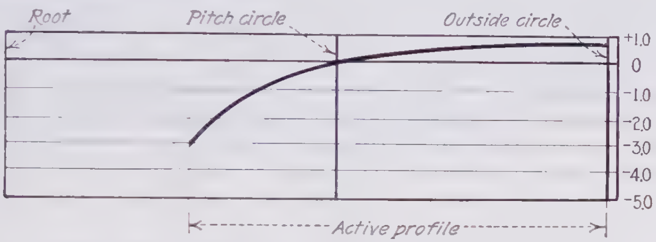


FIG. 71.—Specific sliding between two 40-tooth gears, 14½-deg., variable-center system.

We will now examine the contact conditions when the 10-tooth pinion and the 40-tooth gear are meshed together. We will first determine the pressure angle and the center distance at which they will operate. We have, from Prob. 8, in Chap. III,

$$\operatorname{inv} \alpha_2 = \frac{n(t_1 + T_1) - 2\pi r_1}{2r_1(n + N)} + \operatorname{inv} \alpha_1 \qquad \text{(see Eq. (43))}$$

$$C_2 = \frac{C_1 \cos \alpha_1}{\cos \alpha_2} \qquad \text{(see Eq. (44))}$$

In this example,

r_1 = 14½-deg. pitch radius of pinion = 5.0000 in.
t_1 = tooth thickness of pinion at r_1 = 1.9259 in.
n = number of teeth in pinion = 10
α_1 = pressure angle at r_1 and R_1 = 14½ deg.
R_1 = 14½-deg. pitch radius of gear = 20.0000 in.
T_1 = tooth thickness of gear at R_1 = 1.5708 in.
N = number of teeth in gear = 40
C_1 = center distance for 14½-deg. pressure angle = 25.0000 in.
α_2 = pressure angle of gears when meshed
C_2 = center distance at which gears mesh

whence,

$$\text{inv}\,\alpha_2 = \frac{10(1.92591 + 1.5708) - 31.4160}{10(10 + 40)} + 0.005545$$
$$= 0.012647$$

From the table of involute functions in Chap. III, we get

$$\alpha_2 = 18 \text{ deg. } 58 \text{ min.}$$

whence,

$$\cos \alpha_2 = 0.94571$$
$$C_1 = \frac{25 \times 0.96815}{0.94571} = 25.5931 \text{ in.}$$

The tooth depths will be established as before, whence,

$$1.200 \times \text{working depth} = 25.5931 - (18.80000 + 4.4866)$$
$$= 2.3065 \text{ in.}$$

$$\text{Working depth} = 1.9221 \text{ in.}$$
$$\text{Clearance} = 0.1922 \text{ in.}$$
$$\text{Whole depth} = 2.1143 \text{ in.}$$
$$E_1 = 4.4866 + 2.1143 = 6.6009 \text{ in.} = \text{outside radius of pinion}$$
$$E_2 = 18.80000 + 2.1143 = 20.9143 \text{ in.} = \text{outside radius of gear}$$

We will now examine the contact conditions for this pair of gears. We have, from Eq. (58), in Chap. III,

Number of teeth in contact

$$= \frac{\sqrt{E_1^2 - a^2} + \sqrt{E_2^2 - a_2^2} - C \sin \alpha}{Pn}$$

where,

$$E_1 = 6.6009 \text{ in.}$$
$$a_1 = 4.8407 \text{ in.}$$
$$E_2 = 20.9143 \text{ in.}$$
$$a_2 = 19.3629 \text{ in.}$$
$$C = 25.5931 \text{ in.}$$
$$\alpha = 18 \text{ deg. } 58 \text{ min.}$$
$$\sqrt{E_1^2 - a_1^2} = 4.4876$$
$$\sqrt{E_2^2 - a_2^2} = 7.9046$$
$$C \sin \alpha = 8.3183$$
$$Pn = 3.0415 \text{ in.}$$

whence,

$$\text{contact} = 1.339 \text{ tooth intervals}$$

With favorable sliding conditions, this contact should be sufficient for the transmission of power. We will therefore examine these sliding conditions. We have, from Chap. II,

$$\text{Specific sliding on pinion} = \frac{b_1N_2 - b_2N_1}{b_1N_2}$$

$$\text{Specific sliding on gear} = \frac{b_2N_1 - b_1N_2}{b_2N_1}$$

where b_1 and b_2 are the lengths of the generating lines at the point of contact of the pinion and gear, respectively.

The active profiles of both of these gears extend to the tips of their teeth, whence, the maximum radii of curvature of the active profiles are equal to $\sqrt{E^2 - a^2}$, or 4.4876 in. on the pinion and 7.9046 in. on the gear. By subtracting these figures from the total length of the line of action, we get 0.4137 in. as the minimum radius of curvature of the active profile of the pinion and 3.8307 in. for the gear. Thus, at the beginning of contact, we have

$$b_1 = 0.4137 \text{ in.}$$
$$b_2 = 7.9046 \text{ in.}$$
$$N_1 = 10 \text{ teeth}$$
$$N_2 = 40 \text{ teeth}$$

$$\text{Specific sliding on pinion} = \frac{0.4137 \times 40 - 7.9046 \times 10}{0.4137 \times 40} = -3.84$$

$$\text{Specific sliding on gear} = \frac{7.9046 \times 10 - 0.4137 \times 40}{7.9046 \times 10} = +0.79$$

And at the end of contact, we have

$$b_1 = 4.4876 \text{ in.}$$
$$b_2 = 3.8307 \text{ in.}$$

$$\text{Specific sliding on pinion} = \frac{4.4876 \times 40 - 3.8307 \times 10}{4.4876 \times 40} = +0.78$$

$$\text{Specific sliding on gear} = \frac{3.8307 \times 10 - 4.4876 \times 40}{3.8307 \times 10} = -3.68$$

These specific-sliding conditions are very favorable and are plotted in Fig. 72. It will be noted that these sliding conditions are very similar on both the gear and the pinion, whereas on a pair of gears of standard form with unequal tooth numbers, the sliding conditions on the pinion are much worse than those on the gear. The nature of the sliding action itself is probably of secondary importance and has but little effect on the action

of the gears. The great value of determining this specific sliding lies in the fact that it indicates very clearly the sensitiveness of the active tooth profiles. When the graph of specific sliding has a sharp bend, sensitive tooth profiles are involved, the proper production of which is always difficult. The gears with the greatest specific sliding will always be the most troublesome ones in production. When these specific-sliding conditions are balanced, as in this example, it indicates that the degree of care required in production is about the same for both.

After we have established the largest and the smallest gear of the series with variable tooth proportions, the next step is to

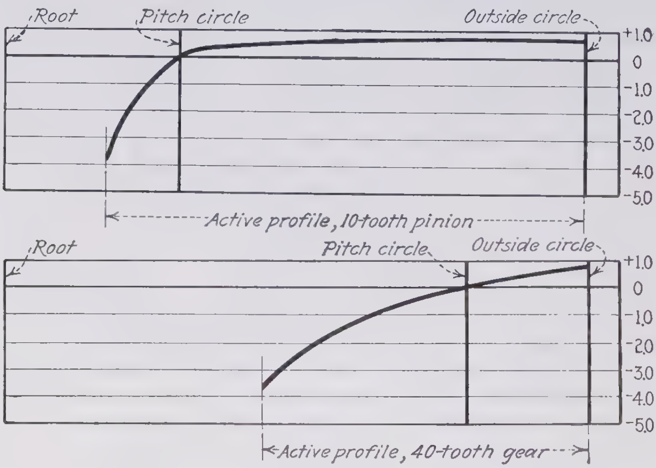


FIG. 72.—Specific sliding between a 10-tooth pinion and a 40-tooth gear, 14½-deg., variable-center system.

establish the intermediate sizes, that is, the gears with more than 10 and less than 40 teeth. We will make all of these gears with a constant root radius and change only their tooth depths to suit the conditions that exist as they are intermeshed among themselves.

The root of the 10-tooth pinion is at the undercut limit, while that of the 40-tooth gear is well outside of it. The simplest method of establishing the intermediate gears would be to make their root radii proportionately larger as their tooth numbers increase, by dividing the difference between the root radius of the 10-tooth pinion and the root radius of the 40-tooth gear by the number of intervals between them and adding this

TABLE XI.—FUNDAMENTAL DIMENSIONS OF 1-D.P. GEARS OF THE 14½-DEG., BASIC-RACK, VARIABLE-CENTER SYSTEM

Number of teeth N	14½-deg. pitch radius R, inches	Radius of base circle, a, inches	Undercut radius A, inches	Root radius H, inches	Tooth thickness at R T, inches
10	5.0	4.84074	4.48655	4.48655	1.92591
11	5.5	5.32481	4.95520	4.96367	1.91407
12	6.0	5.80889	5.42386	5.44078	1.90224
13	6.5	6.29296	5.89251	5.91790	1.89040
14	7.0	6.77703	6.36117	6.39501	1.87856
15	7.5	7.26111	6.82982	6.87213	1.86673
16	8.0	7.74518	7.29848	7.34924	1.85489
17	8.5	8.22925	7.76713	7.82636	1.84305
18	9.0	8.71333	8.23579	8.30347	1.83121
19	9.5	9.19740	8.70444	8.78059	1.81938
20	10.0	9.68148	9.17310	9.25770	1.80754
21	10.5	10.16555	9.64175	9.73482	1.79570
22	11.0	10.64962	10.11041	10.21193	1.78387
23	11.5	11.13370	10.57906	10.68905	1.77203
24	12.0	11.61777	11.04772	11.16616	1.76019
25	12.5	12.10185	11.51637	11.64328	1.74836
26	13.0	12.58592	11.98503	12.12039	1.73652
27	13.5	13.06999	12.45368	12.59751	1.72468
28	14.0	13.55407	12.92234	13.07462	1.71284
29	14.5	14.03814	13.39099	13.55174	1.70101
30	15.0	14.52221	13.85965	14.02885	1.68917
31	15.5	15.00629	14.32830	14.50597	1.67733
32	16.0	15.49036	14.79696	14.98308	1.66550
33	16.5	15.97444	15.26561	15.46020	1.65366
34	17.0	16.45851	15.73427	15.93731	1.64182
35	17.5	16.94258	16.20292	16.41443	1.62999
36	18.0	17.42666	16.67158	16.89154	1.61815
37	18.5	17.91073	17.14023	17.36866	1.60631
38	19.0	18.39481	17.60889	17.84577	1.59447
39	19.5	18.87888	18.07754	18.32289	1.58264
40	20.0	19.36295	18.54620	18.80000	1.57080
41	20.5	19.84703	....	19.3000	1.57080
42	21.0	20.33110	....	19.8000	1.57080
43	21.5	20.81517	....	20.3000	1.57080
44	22.0	21.29925	....	20.8000	1.57080
45	22.5	21.78332	....	21.3000	1.57080
46	23.0	22.26740	....	21.8000	1.57080
47	23.5	22.75147	....	22.3000	1.57080
48	24.0	23.23554	....	22.8000	1.57080
49	24.5	23.71962	....	23.3000	1.57080
50	25.0	24.20369	....	23.8000	1.57080
51	25.5	24.68776	....	24.3000	1.57080
52	26.0	25.17184	....	24.8000	1.57080
53	26.5	25.65591	....	25.3000	1.57080
54	27.0	26.13999	....	25.8000	1.57080
55	27.5	26.62406	....	26.3000	1.57080
56	28.0	27.10813	....	26.8000	1.57080
57	28.5	27.59221	....	27.3000	1.57080
58	29.0	28.07628	....	27.8000	1.57080
59	29.5	28.56036	....	28.3000	1.57080
60	30.0	29.04443	....	28.8000	1.57080

increment to the root radius of the 10-tooth pinion to obtain that of the 11-tooth pinion, and so on, until the series is completed.

This has been done, and the results are given in Table XI, together with other data, such as radii of base circles, radii

TABLE XII.—CENTER DISTANCES, PRESSURE ANGLES, AND TOOTH DEPTHS, 14½-DEG., BASIC-RACK, VARIABLE-CENTER SYSTEM, 1-D.P. GEARS

Number of teeth in gear		Number of teeth in pinion											
		10	11	12	13	14	15	16	17	18	19	20	21
10	Center distance, inches	10.9247											
	Pressure angle	27° 36′											
	Tooth depth, inches	1.7889											
11	Center distance, inches	11.4195	11.9143										
	Pressure angle	27° 7′	26° 40′										
	Tooth depth, inches	1.8052	1.8214										
12	Center distance, inches	11.9143	12.4091	12.9039									
	Pressure angle	26° 40′	26° 14′	25° 49′									
	Tooth depth, inches	1.8214	1.8376	1.8538									
13	Center distance, inches	12.4091	12.9039	13.3987	13.8905								
	Pressure angle	26° 14′	25° 49′	25° 25′	25° 3′								
	Tooth depth, inches	1.8376	1.8538	1.8700	1.8835								
14	Center distance, inches	12.9039	13.3987	13.8905	14.3823	14.8741							
	Pressure angle	25° 49′	25° 25′	25° 3′	24° 41′	24° 20′							
	Tooth depth, inches	1.8538	1.8700	1.8835	1.8970	1.9105							
15	Center distance, inches	13.3987	13.8905	14.3823	14.8741	15.3659	15.8577						
	Pressure angle	25° 25′	25° 3′	24° 41′	24° 20′	24° 0′	23° 41′						
	Tooth depth, inches	1.8700	1.8835	1.8970	1.9105	1.9239	1.9373						
16	Center distance, inches	13.8905	14.3823	14.8741	15.3659	15.8577	16.3473	16.8369					
	Pressure angle	25° 3′	24° 41′	24° 20′	24° 0′	23° 41′	23° 23′	23° 5′					
	Tooth depth, inches	1.8835	1.8970	1.9105	1.9239	1.9373	1.9488	1.9603					
17	Center distance, inches	14.3823	14.8741	15.3659	15.8577	16.3473	16.8369	17.3264	17.8160				
	Pressure angle	24° 41′	24° 20′	24° 0′	23° 41′	23° 23′	23° 5′	22° 48′	22° 31′				
	Tooth depth, inches	1.8970	1.9105	1.9239	1.9373	1.9488	1.9603	1.9717	1.9831				
18	Center distance, inches	14.8741	15.3659	15.8577	16.3473	16.8369	17.3264	17.8160	18.3056	18.7930			
	Pressure angle	24° 20′	24° 0′	23° 41′	23° 23′	23° 5′	22° 48′	22° 31′	22° 15′	21° 59′			
	Tooth depth, inches	1.9105	1.9239	1.9373	1.9488	1.9603	1.9717	1.9831	1.9945	2.0040			

19	Center distance, inches	15.3659	15.8577	16.3473	16.8369	17.3264	17.8160	18.3056	18.7930	19.2805	19.7679		
	Pressure angle	24° 0′	23° 41′	23° 23′	23° 5′	22° 48′	22° 31′	22° 15′	21° 59′	21° 44′	21° 29′		
	Tooth depth, inches	1.9239	1.9373	1.9488	1.9603	1.9717	1.9831	1.9945	2.0040	2.0135	2.0230		
20	Center distance, inches	15.8577	16.3473	16.8369	17.3264	17.8160	18.3056	18.7930	18.2805	19.7679	20.2554	20.7428	
	Pressure angle	23° 41′	23° 23′	23° 5′	22° 48′	22° 31′	22° 15′	21° 59′	21° 44′	21° 29′	21° 15′	21° 1′	
	Tooth depth, inches	1.9373	1.9488	1.9603	1.9717	1.9831	1.9945	2.0040	1.0135	2.0230	2.0324	2.0418	
21	Center distance, inches	16.3473	16.8369	17.3264	17.8160	18.3056	18.7930	19.2805	19.7679	20.2554	20.7428	21.2285	21.7143
	Pressure angle	23° 23′	23° 5′	22° 48′	22° 31′	22° 15′	21° 59′	21° 44′	21° 29′	21° 15′	21° 1′	20° 47′	20° 34′
	Tooth depth, inches	1.9488	1.9603	1.9717	1.9831	1.9945	2.0040	2.0135	2.0230	2.0324	2.0418	2.0497	2.0576
22	Center distance, inches	16.8369	17.3264	17.8160	18.3056	18.7930	19.2805	19.7679	20.2554	20.7428	21.2285	21.7143	22.2000
	Pressure angle	23° 5′	22° 48′	22° 31′	22° 15′	21° 59′	21° 44′	21° 29′	21° 15′	21° 1′	20° 47′	20° 34′	20° 21′
	Tooth depth, inches	1.9603	1.9717	1.9831	1.9945	2.0040	2.0135	2.0230	2.0324	2.0418	2.0497	2.0576	2.0655
23	Center distance, inches	17.3264	17.8160	18.3056	18.7930	19.2805	19.7679	20.2554	20.7428	21.2285	21.7143	22.2000	22.6858
	Pressure angle	22° 48′	22° 31′	22° 15′	21° 59′	21° 44′	21° 29′	21° 15′	21° 1′	20° 47′	20° 34′	20° 21′	20° 8′
	Tooth depth, inches	1.9717	1.9831	1.9945	2.0040	2.0135	2.0230	2.0324	2.0418	2.0497	2.0576	2.0655	2.0734
24	Center distance, inches	17.8160	18.3056	18.7930	19.2805	19.7679	20.2554	20.7428	21.2285	21.7143	22.2000	22.6858	23.1715
	Pressure angle	22° 31′	22° 15′	21° 59′	21° 44′	21° 29′	21° 15′	21° 1′	20° 47′	20° 34′	20° 21′	20° 8′	19° 56′
	Tooth depth, inches	1.9831	1.9945	2.0040	2.0135	2.0230	2.0321	2.0418	2.0497	2.0576	2.0655	2.0734	2.0813
25	Center distance, inches	18.3056	18.7930	19.2805	19.7679	20.2554	20.7428	21.2285	21.7143	22.2000	22.6858	23.1715	23.6558
	Pressure angle	22° 15′	21° 59′	21° 44′	21° 29′	21° 15′	21° 1′	20° 47′	20° 34′	20° 21′	20° 8′	19° 56′	19° 34′
	Tooth depth, inches	1.9945	2.0040	2.0135	2.0230	2.0324	2.0418	2.0497	2.0576	2.0655	2.0734	2.0813	2.0879
26	Center distance, inches	18.7930	19.2805	19.7679	20.2554	20.7428	21.2285	21.7143	22.2000	22.6858	23.1715	23.6558	24.1401
	Pressure angle	21° 59′	21° 44′	21° 29′	21° 15′	21° 1′	20° 47′	20° 34′	20° 21′	20° 8′	19° 56′	19° 44′	19° 32′
	Tooth depth, inches	2.0040	2.0135	2.0230	2.0324	2.0418	2.0497	2.0576	2.0655	2.0734	2.0813	2.0879	2.0945
27	Center distance, inches	19.2805	19.7679	20.2554	20.7428	21.2285	21.7143	22.2000	22.6858	23.1715	23.6558	24.1401	24.6245
	Pressure angle	21° 44′	21° 29′	21° 15′	21° 1′	20° 47′	20° 34′	20° 21′	20° 8′	19° 56′	19° 44′	19° 32′	19° 20′
	Tooth depth, inches	2.0135	2.0230	2.0324	2.0418	2.0497	2.0576	2.0655	2.0734	2.0813	2.0879	2.0945	2.1011
28	Center distance, inches	19.7679	20.2554	20.7428	21.2285	21.7143	22.2000	22.6858	23.1715	23.6558	24.1401	24.6245	25.1088
	Pressure angle	21° 29′	21° 15′	21° 1′	20° 47′	20° 34′	20° 21′	20° 8′	19° 56′	19° 44′	19° 32′	19° 20′	19° 9′
	Tooth depth, inches	2.0230	2.0324	2.0418	2.0497	2.0576	2.0655	2.0734	2.0813	2.0879	2.0945	2.1011	2.1077
29	Center distance, inches	20.2554	20.7428	21.2285	21.7143	22.2000	22.6858	23.1715	23.6558	24.1401	24.6245	25.1088	25.5931
	Pressure angle	21° 15′	21° 1′	20° 47′	20° 34′	20° 21′	20° 8′	19° 56′	19° 44′	19° 32′	19° 20′	19° 9′	18° 58′
	Tooth depth, inches	2.0324	2.0418	2.0497	2.0576	2.0655	2.0734	2.0813	2.0879	2.0945	2.1011	2.1077	2.1143
30	Center distance, inches	20.7428	21.2285	21.7143	22.2000	22.6858	23.1715	23.6558	24.1401	24.6245	25.1088	25.5931	26.0759
	Pressure angle	21° 1′	20° 47′	20° 34′	20° 21′	20° 8′	19° 56′	19° 44′	19° 32′	19° 20′	19° 9′	18° 58′	18° 47′
	Tooth depth, inches	2.0418	2.0497	2.0576	2.0655	2.0734	2.0813	2.0879	2.0945	2.1011	2.1077	2.1143	2.1196

TABLE XII.—CENTER DISTANCES, PRESSURE ANGLES, AND TOOTH DEPTHS. 14½-DEG., BASIC-RACK, VARIABLE-CENTER SYSTEM, 1-D.P. GEARS (*Continued*)

Number of teeth in gear		Number of teeth in pinion											
		10	11	12	13	14	15	16	17	18	19	20	21
31	Center distance, inches	21.2285	21.7143	22.2000	22.6858	23.1715	23.6558	24.1401	24.6245	25.1088	25.5931	26.0759	26.5588
	Pressure angle	20° 47′	20° 34′	20° 21′	20° 8′	19° 56′	19° 44′	19° 32′	19° 20′	19° 9′	18° 58′	18° 47′	18° 36′
	Tooth depth, inches	2.0497	2.0576	2.0655	2.0734	2.0813	2.0879	2.0945	2.1011	2.1077	2.1143	2.1196	2.1249
32	Center distance, inches	21.7143	22.2000	22.6858	23.1715	23.6558	24.1401	24.6245	25.1088	25.5931	26.0759	26.5588	27.0416
	Pressure angle	20° 34′	20° 21′	20° 8′	19° 56′	19° 44′	19° 32′	19° 20′	19° 9′	18° 58′	18° 47′	18° 36′	18° 26′
	Tooth depth, inches	2.0576	2.0655	2.0734	2.0813	2.0879	2.0945	2.1011	2.1077	2.1143	2.1196	2.1249	2.1302
33	Center distance, inches	22.2000	22.6858	23.1715	23.6558	24.1401	24.6245	25.1088	25.5931	26.0759	26.5588	27.0416	27.5245
	Pressure angle	20° 21′	20° 8′	19° 56′	19° 44′	19° 32′	19° 20′	19° 9′	18° 58′	18° 47′	18° 36′	18° 26′	18° 16′
	Tooth depth, inches	2.0655	2.0734	2.0813	2.0879	2.0945	2.1011	2.1077	2.1143	2.1196	2.1249	2.1302	2.1354
34	Center distance, inches	22.6858	23.1715	23.6558	24.1401	24.6245	25.1088	25.5931	26.0759	26.5588	27.0416	27.5245	28.0073
	Pressure angle	20° 8′	19° 56′	19° 44′	19° 32′	19° 20′	19° 9′	18° 58′	18° 47′	18° 36′	18° 26′	18° 16′	18° 5′
	Tooth depth, inches	2.0734	2.0813	2.0879	2.0945	2.1011	2.1077	2.1143	2.1196	2.1249	2.1302	2.1354	2.1406
35	Center distance, inches	23.1715	23.6558	24.1401	24.6245	25.1088	25.5931	26.0759	26.5588	27.0416	27.5245	28.0073	28.4894
	Pressure angle	19° 56′	19° 44′	19° 32′	19° 20′	19° 9′	18° 58′	18° 47′	18° 36′	18° 26′	18° 16′	18° 5′	17° 55′
	Tooth depth, inches	2.0813	2.0879	2.0945	2.1011	2.1077	2.1143	2.1196	2.1249	2.1302	2.1354	2.1406	2.1452
36	Center distance, inches	23.6558	24.1401	24.6245	25.1088	25.5931	26.0759	26.5588	27.0416	27.5245	28.0073	28.4894	28.9715
	Pressure angle	19° 44′	19° 32′	19° 20′	19° 9′	18° 58′	18° 47′	18° 36′	18° 26′	18° 16′	18° 5′	17° 55′	17° 46′
	Tooth depth, inches	2.0879	2.0945	2.1011	2.1077	2.1143	2.1196	2.1249	2.1302	2.1354	2.1406	2.1452	2.1498
37	Center distance, inches	24.1401	24.6245	25.1088	25.5931	26.0759	26.5588	27.0416	27.5245	28.0073	28.4894	28.9715	29.4536
	Pressure angle	19° 32′	19° 20′	19° 9′	18° 58′	18° 47′	18° 36′	18° 26′	18° 16′	18° 5′	17° 55′	17° 46′	17° 36′
	Tooth depth, inches	2.0945	2.1011	2.1077	2.1143	2.1196	2.1249	2.1302	2.1354	2.1406	2.1452	2.1498	2.1544
38	Center distance, inches	24.6245	25.1088	25.5931	26.0759	26.5588	27.0416	27.5245	28.0073	28.4894	28.9715	29.4536	29.9357
	Pressure angle	19° 20′	19° 9′	18° 58′	18° 47′	18° 36′	18° 26′	18° 16′	18° 5′	17° 55′	17° 46′	17° 36′	17° 26′
	Tooth depth, inches	2.1011	2.1077	2.1143	2.1196	2.1249	2.1302	2.1354	2.1406	2.1452	2.1498	2.1544	2.1590
39	Center distance, inches	25.1088	25.5931	26.0759	25.5588	27.0416	27.5245	28.0073	28.4894	28.9715	29.4536	29.9357	30.4178
	Pressure angle	19° 9′	18° 58′	18° 47′	18° 36′	18° 26′	18° 16′	18° 5′	17° 55′	17° 46′	17° 36′	17° 26′	17° 17′
	Tooth depth, inches	2.1077	2.1143	2.1196	2.1249	2.1302	2.1354	2.1406	2.1452	2.1498	2.1544	2.1590	2.1635

40	Center distance, inches	25.5931	26.0759	26.5588	27.0416	27.5245	28.0073	28.4894	28.9715	29.4536	29.9357	30.4178	30.8986
	Pressure angle	18° 58′	18° 47′	18° 36′	18° 26′	18° 16′	18° 5′	17° 55′	17° 46′	17° 36′	17° 26′	17° 17′	17° 8′
	Tooth depth, inches	2.1143	2.1196	2.1249	2.1302	2.1354	2.1406	2.1452	2.1498	2.1544	2.1590	2.1635	2.1669
41	Center distance, inches	26.0941	26.5765	27.0589	27.5413	28.0237	28.5062	28.9886	29.4710	29.9531	30.4358	30.9182	31.3983
	Pressure angle	18° 54′	18° 44′	18° 34′	18° 21′	18° 11′	18° 5′	17° 51′	17° 44′	17° 34′	17° 24′	17° 15′	17° 6′
	Tooth depth, inches	2.1153	2.1205	2.1258	2.1310	2.1361	2.1412	2.1457	2.1502	2.1547	2.1591	2.1636	2.1670
42	Center distance, inches	26.5952	27.0776	27.5599	28.0423	28.5246	29.0070	29.4893	29.9717	30.4540	30.9364	31.4187	31.8988
	Pressure angle	18° 50′	18° 40′	18° 30′	18° 20′	18° 10′	18° 0′	17° 50′	17° 40′	17° 31′	17° 22′	17° 13′	17° 4′
	Tooth depth, inches	2.1162	2.1210	2.1264	2.1313	2.1364	2.1414	2.1459	2.1505	2.1551	2.1595	2.1643	2.1671
43	Center distance, inches	27.0962	27.5785	28.0608	28.5431	29.0254	29.5077	29.9899	30.4722	30.9545	31.4368	31.9191	32.3992
	Pressure angle	18° 46′	18° 36′	18° 26′	18° 16′	18° 6′	17° 56′	17° 47′	17° 38′	17° 29′	17° 20′	17° 11′	17° 2′
	Tooth depth, inches	2.1172	2.1220	2.1267	2.1315	2.1366	2.1415	2.1461	2.1508	2.1554	2.1600	2.1647	2.1675
44	Center distance, inches	27.5972	28.0794	28.5617	29.0139	29.5262	30.0084	30.4906	30.9729	31.4551	31.9374	32.4196	32.9996
	Pressure angle	18° 42′	18° 32′	18° 22′	18° 12′	18° 3′	17° 54′	17° 45′	17° 36′	17° 27′	17° 18′	17° 9′	17° 0′
	Tooth depth, inches	2.1181	2.1228	2.1275	2.1322	2.1369	2.1416	2.1463	2.1510	2.1557	2.1604	2.1651	2.1679
45	Center distance, inches	28.0983	28.5805	29.0626	29.5448	30.0270	30.5092	30.9913	31.4735	31.9557	32.4378	32.9200	33.4000
	Pressure angle	18° 39′	18° 29′	18° 19′	18° 9′	18° 0′	17° 51′	17° 42′	17° 33′	17° 24′	17° 15′	17° 6′	16° 58′
	Tooth depth, inches	2.1191	2.1237	2.1284	2.1330	2.1377	2.1423	2.1469	2.1516	2.1562	2.1609	2.1655	2.1683
46	Center distance, inches	28.5993	29.0814	29.5635	30.0456	30.5277	31.0099	31.4920	31.9741	32.4562	32.9383	33.4204	33.9004
	Pressure angle	18° 35′	18° 25′	18° 16′	18° 7′	17° 58′	17° 49′	17° 40′	17° 31′	17° 22′	17° 13′	17° 4′	16° 56′
	Tooth depth, inches	2.1200	2.1246	2.1292	2.1338	2.1384	2.1430	2.1475	2.1521	2.1567	2.1613	2.1659	2.1687
47	Center distance, inches	29.1003	29.5824	30.0644	30.5465	31.0285	31.5106	31.9927	32.4747	32.9568	33.4388	33.9209	34.4008
	Pressure angle	18° 32′	18° 23′	18° 14′	18° 5′	17° 56′	17° 47′	17° 38′	17° 29′	17° 20′	17° 11′	17° 2′	16° 54′
	Tooth depth, inches	2.1210	2.1255	2.1301	2.1346	2.1391	2.1437	2.1482	2.1527	2.1572	2.1618	2.1663	2 1691
48	Center distance, inches	29.6013	30.0833	30.5653	31.0473	31.5293	32.0113	32.4933	32.9753	33.4573	33.9393	34.4213	34.9012
	Pressure angle	18° 29′	18° 20′	18° 11′	18° 2′	17° 53′	17° 44′	17° 35′	17° 26′	17° 17′	17° 8′	17° 0′	16° 52′
	Tooth depth, inches	2.1219	2.1264	2.1309	2.1353	2.1398	2.1443	2.1488	2.1533	2.1577	2.1622	2.1667	2.1695
49	Center distance, inches	30.1024	30.5843	31.0663	31.5482	32.0302	32.5121	32.9940	33.4760	33.9579	34.4399	34.9218	35.4016
	Pressure angle	18° 26′	18° 17′	18° 8′	17° 59′	17° 50′	17° 41′	17° 32′	17° 23′	17° 14′	17° 6′	16° 58′	16° 50′
	Tooth depth, inches	2.1229	2.1273	2.1317	2.1362	2.1406	2.1450	2.1494	2.1538	2.1583	2.1627	2.1671	2.1698
50	Center distance, inches	30.6034	31.0853	31.5672	32.0490	32.5309	33.0128	33.4947	33.9766	34.4584	34.9403	35.4222	35.9020
	Pressure angle	18° 22′	18° 13′	18° 4′	17° 55′	17° 46′	17° 37′	17° 28′	17° 20′	17° 12′	17° 4′	16° 56′	16° 48′
	Tooth depth, inches	2.1238	2.1282	2.1325	2.1369	2.1413	2.1457	2.1500	2.1544	2.1588	2.1631	2.1675	2.1701
52	Center distance, inches	31.6052	32.0870	32.5688	33.0505	33.5323	34.0141	34.4959	34.9777	35.4594	35.9412	36.4230	36.9028
	Pressure angle	18° 16′	18° 7′	17° 59′	17° 50′	17° 41′	17° 32′	17° 24′	17° 16′	17° 8′	17° 0′	16° 53′	16° 45′
	Tooth depth, inches	2.1254	2.1298	2.1340	2.1383	2.1426	2.1469	2.1511	2.1554	2.1597	2.1639	2.1682	2.1708

TABLE XII.—CENTER DISTANCES, PRESSURE ANGLES, AND TOOTH DEPTHS, 14½-DEG., BASIC-RACK, VARIABLE-CENTER SYSTEM, 1-D.P. GEARS (*Continued*)

Number of teeth in gear		Number of teeth in pinion											
		10	11	12	13	14	15	16	17	18	19	20	21
54	Center distance, inches	32.6070	33.0887	33.5704	34.0520	34.5337	35.0154	35.4971	35.9788	36.4604	36.9421	37.4237	37.9035
	Pressure angle	18° 11′	18° 2′	17° 54′	17° 45′	17° 36′	17° 28′	17° 20′	17° 12′	17° 4′	16° 57′	16° 50′	16° 42′
	Tooth depth, inches	2.1271	2.1314	2.1355	2.1397	2.1439	2.1481	2.1522	2.1564	2.1606	2.1647	2.1689	2.1715
56	Center distance, inches	33.6088	34.0904	34.5720	35.0535	35.5351	36.0167	36.4983	36.9799	37.4614	37.9430	38.4245	38.9042
	Pressure angle	18° 5′	17° 57′	17° 49′	17° 40′	17° 31′	17° 24′	17° 16′	17° 8′	17° 1′	16° 54′	16° 47′	16° 39′
	Tooth depth, inches	2.1287	2.1329	2.1370	2.1411	2.1452	2.1493	2.1533	2.1574	2.1615	2.1655	2.1696	2.1722
58	Center distance, inches	34.6106	35.0921	35.5736	36.0550	36.5365	37.0180	37.4995	37.9809	38.4624	38.9438	39.4252	39.9049
	Pressure angle	18° 0′	17° 52′	17° 44′	17° 35′	17° 27′	17° 20′	17° 12′	17° 5′	16° 58′	16° 51′	16° 44′	16° 36′
	Tooth depth, inches	2.1304	2.1344	2.1384	2.1424	2.1464	2.1504	2.1544	2.1584	2.1624	2.1663	2.1703	2.1728
60	Center distance, inches	35.6124	36.0938	36.5751	37.0565	37.5378	38.0192	38.5006	38.9819	39.4633	39.9446	40.4260	40.9056
	Pressure angle	17° 55′	17° 47′	17° 39′	17° 31′	17° 23′	17° 16′	17° 9′	17° 2′	16° 55′	16° 48′	16° 41′	16° 34′
	Tooth depth, inches	2.1320	2.1359	2.1398	2.1437	2.1476	2.1515	2.1554	2.1593	2.1632	2.1671	2.1710	2.1734
		22	23	24	25	26	27	28	29	30	31	32	33
22	Center distance, inches	22.6858											
	Pressure angle	20° 8′											
	Tooth depth, inches	2.0734											
23	Center distance, inches	23.1715	23.6558										
	Pressure angle	19° 56′	19° 44′										
	Tooth depth, inches	2.0813	2.0879										
24	Center distance, inches	23.6558	24.1401	24.6245									
	Pressure angle	19° 44′	19° 32′	19° 20′									
	Tooth depth, inches	2.0879	2.0945	2.1011									
25	Center distance, inches	24.1401	24.6245	25.1088	25.5931								
	Pressure angle	19° 32′	19° 20′	19° 9′	18° 58′								
	Tooth depth, inches	2.0945	2.1011	2.1077	2.1143								

26	Center distance, inches	24.6245	25.1088	25.5931	26.0759	26.5588							
	Pressure angle	19° 20′	19° 9′	18° 58′	18° 47′	18° 36′							
	Tooth depth, inches	2.1011	2.1077	2.1143	2.1196	2.1249							
27	Center distance, inches	25.1088	25.5931	26.0759	26.5588	27.0416	27.5245						
	Pressure angle	19° 9′	18° 58′	18° 47′	18° 36′	18° 26′	18° 16′						
	Tooth depth, inches	2.1077	2.1143	2.1196	2.1249	2.1302	2.1354						
28	Center distance, inches	25.5931	26.0759	26.5588	27.0416	27.5245	28.0073	28.4894					
	Pressure angle	18° 58′	18° 47′	18° 36′	18° 26′	18° 16′	18° 5′	17° 55′					
	Tooth depth, inches	2.1143	2.1196	2.1249	2.1302	2.1354	2.1406	2.1452					
29	Center distance, inches	26.0759	26.5588	27.0416	27.5245	28.0073	28.4894	28.9715	29.4536				
	Pressure angle	18° 47′	18° 36′	18° 26′	18° 16′	18° 5′	17° 55′	17° 46′	17° 36′				
	Tooth depth, inches	2.1196	2.1249	2.1302	2.1354	2.1406	2.1452	2.1498	2.1544				
30	Center distance, inches	26.5588	27.0416	27.5245	28.0073	28.4894	28.9715	29.4536	29.9357	30.4178			
	Pressure angle	18° 36′	18° 26′	18° 16′	18° 5′	17° 55′	17° 46′	17° 36′	17° 26′	17° 17′			
	Tooth depth, inches	2.1249	2.1302	2.1354	2.1406	2.1452	2.1498	2.1544	2.1590	2.1635			
31	Center distance, inches	27.0416	27.5245	28.0073	28.4894	28.9715	29.4536	29.9357	30.4178	30.8986	31.3795		
	Pressure angle	18° 26′	18° 16′	18° 5′	17° 55′	17° 46′	17° 36′	17° 26′	17° 17′	17° 8′	16° 59′		
	Tooth depth, inches	2.1302	2.1354	2.1406	2.1452	2.1498	2.1544	2.1590	2.1635	2.1669	2.1703		
32	Center distance, inches	27.5245	28.0073	28.4894	28.9715	29.4536	29.9357	30.4178	30.8986	31.3795	31.8603	32.3412	
	Pressure angle	18° 16′	18° 5′	17° 55′	17° 46′	17° 36′	17° 26′	17° 17′	17° 8′	16° 59′	16° 50′	16° 41′	
	Tooth depth, inches	2.1354	2.1406	2.1452	2.1498	2.1544	2.1590	2.1635	2.1669	2.1703	2.1737	2.1771	
33	Center distance, inches	28.0073	28.4894	28.9715	29.4536	29.9357	30.4178	30.8986	31.3795	31.8603	32.3412	32.8220	33.3013
	Pressure angle	18° 5′	17° 55′	17° 46′	17° 36′	17° 26′	17° 17′	17° 8′	16° 59′	16° 50′	16° 41′	16° 32′	16° 23′
	Tooth depth, inches	2.1406	2.1452	2.1498	2.1544	2.1590	2.1635	2.1669	2.1703	2.1737	2.1771	2.1805	2.1827
34	Center distance, inches	28.4894	28.9715	29.4536	29.9357	30.4178	30.8986	31.3795	31.8603	32.3412	32.8220	33.3013	33.7806
	Pressure angle	17° 55′	17° 46′	17° 36′	17° 26′	17° 17′	17° 8′	16° 59′	16° 50′	16° 41′	16° 32′	16° 23′	16° 14′
	Tooth depth, inches	2.1452	2.1498	2.1544	2.1590	2.1635	2.1669	2.1703	2.1737	2.1771	2.1805	2.1827	2.1849
35	Center distance, inches	28.9715	29.4536	29.9357	30.4178	30.8986	31.3795	31.8603	32.3412	32.8220	33.3013	33.7806	34.2599
	Pressure angle	17° 46′	17° 36′	17° 26′	17° 17′	17° 8′	16° 59′	16° 50′	16° 41′	16° 32′	16° 23′	16° 14′	16° 6′
	Tooth depth, inches	2.1498	2.1544	2.1590	2.1635	2.1669	2.1703	2.1737	2.1771	2.1805	2.1827	2.1849	2.1871
36	Center distance, inches	29.4536	29.9357	30.4178	30.8986	31.3795	31.8603	32.3412	32.8220	33.3013	33.7806	34.2599	34.7392
	Pressure angle	17° 36′	17° 26′	17° 17′	17° 8′	16° 59′	16° 50′	16° 41′	16° 32′	16° 23′	16° 14′	16° 6′	15° 58′
	Tooth depth, inches	2.1544	2.1590	2.1635	2.1669	2.1703	2.1737	2.1771	2.1805	2.1827	2.1849	2.1871	2.1893
37	Center distance, inches	29.9357	30.4178	30.8986	31.3795	31.8603	32.3412	32.8220	33.3013	33.7806	34.2599	34.7392	35.2185
	Pressure angle	17° 26′	17° 17′	17° 8′	16° 59′	16° 50′	16° 41′	16° 32′	16° 23′	16° 14′	16° 6′	15° 58′	15° 49′
	Tooth depth, inches	2.1590	2.1635	2.1669	2.1703	2.1737	2.1771	2.1805	2.1827	2.1849	2.1871	2.1893	2.1915

TABLE XII.—CENTER DISTANCES, PRESSURE ANGLES, AND TOOTH DEPTHS, 14½-DEG., BASIC-RACK, VARIABLE-CENTER SYSTEM, 1-D.P. GEARS (*Continued*)

Number of teeth in gear		Number of teeth in pinion											
		22	23	24	25	26	27	28	29	30	31	32	33
38	Center distance, inches	30.4178	30.8986	31.3795	31.8603	32.3412	32.8220	33.3013	33.7806	34.2599	34.7392	35.2185	35.6974
	Pressure angle	17° 17′	17° 8′	16° 59′	16° 50′	16° 41′	16° 32′	16° 23′	16° 14′	16° 6′	15° 58′	15° 49′	15° 41′
	Tooth depth, inches	2.1635	2.1669	2.1703	2.1737	2.1771	2.1805	2.1827	2.1849	2.1871	2.1893	2.1915	2.1928
39	Center distance, inches	30.8986	31.3795	31.8603	32.3412	32.8220	33.3013	33.7806	34.2599	34.7392	35.2185	35.6974	36.1763
	Pressure angle	17° 8′	16° 59′	16° 50′	16° 41′	16° 32′	16° 23′	16° 14′	16° 6′	15° 58′	15° 49′	15° 41′	15° 33′
	Tooth depth, inches	2.1669	2.1703	2.1737	2.1771	2.1805	2.1827	2.1849	2.1871	2.1893	2.1915	2.1928	2.1941
40	Center distance, inches	31.3795	31.8603	32.3412	32.8220	33.3013	33.7806	34.2599	34.7392	35.2185	35.6974	36.1763	36.6552
	Pressure angle	16° 59′	16° 50′	16° 41′	16° 32′	16° 23′	16° 14′	16° 6′	15° 58′	15° 49′	15° 41′	15° 33′	15° 25′
	Tooth depth, inches	2.1703	2.1737	2.1771	2.1805	2.1827	2.1849	2.1871	2.1893	2.1915	2.1928	2.1941	2.1954
41	Center distance, inches	31.8798	32.3605	32.8413	33.3221	33.8014	34.2807	34.7600	35.2394	35.7187	36.1976	36.6764	37.1553
	Pressure angle	16° 57′	16° 48′	16° 39′	16° 30′	16° 21′	16° 13′	16° 5′	15° 56′	15° 48′	15° 40′	15° 32′	15° 24′
	Tooth depth, inches	2.1706	2.1739	2.1772	2.1806	2.1828	2.1850	2.1872	2.1894	2.1917	2.1929	2.1941	2.1954
42	Center distance, inches	32.3801	32.8607	33.3414	33.8222	34.3015	34.7808	35.2601	35.7396	36.2189	36.6977	37.1765	37.6554
	Pressure angle	16° 55′	16° 46′	16° 37′	16° 28′	16° 19′	16° 11′	16° 4′	15° 55′	15° 47′	15° 39′	15° 31′	15° 24′
	Tooth depth, inches	2.1709	2.1741	2.1773	2.1807	2.1829	2.1851	2.1873	2.1896	2.1918	2.1930	2.1942	2.1955
43	Center distance, inches	32.8804	33.3609	33.8415	34.3223	34.8016	35.2809	35.7602	36.2397	36.7191	37.1978	37.6766	38.1555
	Pressure angle	16° 53′	16° 44′	16° 35′	16° 26′	16° 17′	16° 10′	16° 2′	15° 54′	15° 46′	15° 39′	15° 31′	15° 23′
	Tooth depth, inches	2.1712	2.1743	2.1774	2.1808	2.1830	2.1852	2.1874	2.1897	2.1920	2.1931	2.1942	2.1955
44	Center distance, inches	33.3807	33.8611	34.3416	34.8224	35.3017	35.7810	36.2603	36.7399	37.2193	37.6979	38.1767	38.6556
	Pressure angle	16° 51′	16° 42′	16° 33′	16° 24′	16° 16′	16° 8′	16° 1′	15° 53′	15° 45′	15° 38′	15° 30′	15° 22′
	Tooth depth, inches	2.1715	2.1745	2.1775	2.1809	2.1831	2.1853	2.1875	2.1899	2.1921	2.1932	2.1943	2.1956
45	Center distance, inches	33.8809	34.3612	34.8417	35.3225	35.8018	36.2811	36.7604	37.2400	37.7195	38.1980	38.6768	39.1556
	Pressure angle	16° 49′	16° 40′	16° 31′	16° 22′	16° 14′	16° 7′	16° 0′	15° 52′	15° 44′	15° 37′	15° 29′	15° 22′
	Tooth depth, inches	2.1717	2.1747	2.1776	2.1810	2.1832	2.1854	2.1876	2.1900	2.1923	2.1933	2.1943	2.1956
46	Center distance, inches	34.3811	34.8613	35.3418	35.8226	36.3019	36.7812	37.2605	37.7402	38.2197	38.6981	39.1769	39.6557
	Pressure angle	16° 47′	16° 38′	16° 29′	16° 20′	16° 13′	16° 5′	15° 58′	15° 51′	15° 43′	15° 36′	15° 28′	15° 21′
	Tooth depth, inches	2.1719	2.1749	2.1777	2.1811	2.1833	2.1855	2.1877	2.1901	2.1924	2.1934	2.1944	2.1957

47	Center distance, inches	34.8813	35.3614	35.8419	36.3227	36.8020	37.2813	37.7606	38.2403	38.7199	38.1982	39.6770	40.1558
	Pressure angle	16° 45′	16° 36′	16° 28′	16° 19′	16° 11′	16° 4′	15° 57′	15° 50′	15° 42′	15° 35′	15° 28′	15° 20′
	Tooth depth, inches	2.1721	2.1750	2.1778	2.1812	2.1834	2.1856	2.1878	2.1902	2.1926	2.1935	2.1944	2.1957
48	Center distance, inches	35.3815	35.8615	36.3420	36.8228	37.3021	37.7814	38.2607	38.7405	39.2201	39.6983	40.1771	40.6559
	Pressure angle	16° 43′	16° 34′	16° 26′	16° 18′	16° 10′	16° 3′	15° 56′	15° 49′	15° 42′	15° 34′	15° 27′	15° 20′
	Tooth depth, inches	2.1723	2.1751	2.1779	2.1813	2.1835	2.1857	2.1879	2.1903	2.1927	2.1936	2.1945	2.1958
49	Center distance, inches	35.8817	36.3616	36.8421	37.3229	37.8022	38.2815	38.7608	39.2406	39.7203	40.1984	40.6772	41.1560
	Pressure angle	16° 41′	16° 33′	16° 25′	16° 17′	16° 9′	16° 2′	15° 55′	15° 48′	15° 41′	15° 33′	15° 26′	15° 19′
	Tooth depth, inches	2.1725	2.1752	2.1780	2.1814	2.1836	2.1858	2.1880	2.1904	2.1929	2.1937	2.1945	2.1958
50	Center distance, inches	36.3819	36.8617	37.3422	37.8230	38.3023	38.7816	39.2609	39.7407	40.2205	40.6985	41.1773	41.6561
	Pressure angle	16° 40′	16° 32′	16° 24′	16° 16′	16° 8′	16° 1′	15° 54′	15° 47′	15° 40′	15° 33′	15° 26′	15° 19′
	Tooth depth, inches	2.1727	2.1753	2.1781	2.1815	2.1837	2.1859	2.1881	2.1905	2.1930	2.1938	2.1946	2.1959
52	Center distance, inches	37.3826	37.8624	38.3426	38.8232	39.3025	39.7819	40.2613	40.7411	41.2209	41.6988	42.1776	42.6563
	Pressure angle	16° 37′	16° 29′	16° 21′	16° 14′	16° 6′	15° 59′	15° 52′	15° 45′	15° 38′	15° 31′	15° 25′	15° 18′
	Tooth depth, inches	2.1734	2.1759	2.1786	2.1817	2.1839	2.1862	2.1884	2.1908	2.1933	2.1940	2.1949	2.1961
54	Center distance, inches	38.3833	38.8631	39.3430	39.8233	40.3027	40.7822	41.2618	41.7415	42.2213	42.6991	43.1779	43.6565
	Pressure angle	16° 34′	16° 26′	16° 19′	16° 12′	16° 4′	15° 57′	15° 50′	15° 43′	15° 36′	15° 30′	15° 24′	15° 17′
	Tooth depth, inches	2.1740	2.1765	2.1791	2.1819	2.1841	2.1865	2.1888	2.1911	2.1936	2.1943	2.1951	2.1963
56	Center distance, inches	39.3840	39.8637	40.3433	40.8234	41.3029	41.7825	42.2622	42.7419	43.2216	43.6994	44.1782	44.6567
	Pressure angle	16° 31′	16° 24′	16° 17′	16° 10′	16° 2′	15° 55′	15° 48′	15° 41′	15° 35′	15° 29′	15° 23′	15° 16′
	Tooth depth, inches	2.1746	2.1771	2.1795	2.1821	2.1843	2.1867	2.1891	2.1914	2.1939	2.1946	2.1953	2.1965
58	Center distance, inches	40.3846	40.8643	41.3439	41.8236	42.3032	42.7829	43.2626	43.7423	44.2219	44.6997	45.1786	45.6569
	Pressure angle	16° 29′	16° 22′	16° 15′	16° 8′	16° 0′	15° 53′	15° 46′	15° 40′	15° 34′	15° 28′	15° 22′	15° 15′
	Tooth depth, inches	2.1752	2.1777	2.1800	2.1824	2.1847	2.1871	2.1894	2.1917	2.1941	2.1948	2.1955	2.1967
60	Center distance, inches	41.3852	41.8649	42.3445	42.8241	43.3037	43.7833	44.2630	44.7426	45.2222	45.7000	46.1786	46.6571
	Pressure angle	16° 27′	16° 20′	16° 13′	16° 6′	15° 59′	15° 52′	15° 45′	15° 39′	15° 33′	15° 27′	15° 21′	15° 15′
	Tooth depth, inches	2.1758	2.1782	2.1805	2.1828	2.1851	2.1874	1.1897	2.1920	2.1943	2.1950	2.1957	2.1969
		34	35	36	37	38	39	40	41	42	43	44	45
34	Center distance, inches	34.2599											
	Pressure angle	16° 6′											
	Tooth depth, inches	2.1871											
35	Center distance, inches	34.7342	35.2185										
	Pressure angle	15° 58′	15° 49′										
	Tooth depth, inches	2.1893	2.1915										

TABLE XII.—CENTER DISTANCES, PRESSURE ANGLES, AND TOOTH DEPTHS, 14½-DEG., BASIC-RACK, VARIABLE-CENTER SYSTEM, 1-D.P. GEARS (*Continued*)

Number of teeth in gear		Number of teeth in pinion: 34	35	36	37	38	39	40	41	42	43	44	45
36	**Center distance, inches**	35.2185	35.6974	36.1763									
	Pressure angle	15° 49′	15° 41′	15° 33′									
	Tooth depth, inches	2.1915	2.1928	2.1941									
37	Center distance, inches	35.6974	36.1763	36.6652	37.1341								
	Pressure angle	15° 41′	15° 33′	15° 25′	15° 17′								
	Tooth depth, inches	2.1928	2.1941	2.1954	2.1967								
38	Center distance, inches	36.1763	36.6552	37.1341	37.6130	38.0904							
	Pressure angle	15° 33′	15° 25′	15° 17′	15° 9′	15° 1′							
	Tooth depth, inches	2.1941	2.1954	2.1967	2.1980	2.1984							
39	Center distance, inches	36.6552	37.1341	37.6130	38.0904	38.5678	39.0452						
	Pressure angle	15° 25′	15° 17′	15° 9′	15° 1′	14° 53′	14° 45′						
	Tooth depth, inches	2.1954	2.1967	2.1980	2.1984	2.1988	2.1992						
40	Center distance, inches	37.1341	37.6130	38.0904	38.5678	39.0452	39.5226	40.0000					
	Pressure angle	15° 17′	15° 9′	15° 1′	14° 53′	14° 45′	14° 37′	14° 30′					
	Tooth depth, inches	2.1967	2.1980	2.1984	2.1988	2.1992	2.1996	2.2000					
41	Center distance, inches	37.6342	38.1130	38.5904	39.0678	39.5452	40.0226	40.5000	41.0000				
	Pressure angle	15° 17′	15° 9′	15° 1′	14° 53′	14° 45′	14° 37′	14° 30′	14° 30′				
	Tooth depth, inches	2.1967	2.1980	2.1984	2.1988	2.1992	2.1996	2.2000	2.2000				
42	Center distance, inches	38.1342	38.6131	39.0905	39.5678	40.0452	40.5226	41.0000	41.5000	42.0000			
	Pressure angle	15° 16′	15° 8′	15° 1′	14° 53′	14° 45′	14° 37′	14° 30′	14° 30′	14° 30′			
	Tooth depth, inches	2.1968	2.1980	2.1984	2.1988	2.1992	2.1996	2.2000	2.2000	2.2000			
43	Center distance, inches	38.6343	39.1131	39.5905	40.0679	40.5452	41.0226	41.5000	42.0000	42.5000	43.0000		
	Pressure angle	15° 16′	15° 8′	15° 0′	14° 53′	14° 45′	14° 37′	14° 30′	14° 30′	14° 30′	14° 30′		
	Tooth depth, inches	2.1969	2.1981	2.1984	2.1988	2.1992	2.1996	2.2000	2.2000	2.2000	2.2000		
44	Center distance, inches	39.1344	39.6132	40.0900	40.5679	41.0452	41.5226	42.0000	42.5000	43.0000	43.5000	44.0000	
	Pressure angle	15° 15′	15° 7′	15° 0′	14° 53′	14° 45′	14° 37′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′	
	Tooth depth, inches	2.1969	2.1981	2.1985	2.1988	2.1992	2.1996	2.2000	2.2000	2.2000	2.2000	2.2000	

45	Center distance, inches	39.6344	40.1132	40.5900	41.0679	41.5452	42.0226	42.5000	43.0000	43.5000	44.0000	44.5000	45.0000
	Pressure angle	15° 15′	15° 7′	15° 0′	14° 52′	14° 45′	14° 37′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′
	Tooth depth, inches	2.1970	2.1981	2.1985	2.1988	2.1992	2.1996	2.2000	2.2000	2.2000	2.2000	2.2000	2.2000
46	Center distance, inches	40.1345	40.6133	41.0906	41.5680	42.0453	42.5226	43.0000	43.5000	44.0000	44.5000	45.0000	45.5000
	Pressure angle	15° 14′	15° 6′	14° 59′	14° 52′	14° 45′	14° 37′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′
	Tooth depth, inches	2.1970	2.1982	2.1985	2.1988	2.1992	2.1996	2.2000	2.2000	2.2000	2.2000	2.2000	2.2000
47	Center distance, inches	40.6345	41.1133	41.5907	42.0680	42.5453	43.0226	43.5000	44.0000	44.5000	45.0000	45.5000	46.0000
	Pressure angle	15° 14′	15° 6′	14° 59′	14° 52′	14° 45′	14° 37′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′
	Tooth depth, inches	2.1970	2.1982	2.1986	2.1989	2.1992	2.1996	2.2000	2.2000	2.2000	2.2000	2.2000	2.2000
48	Center distance, inches	41.1346	41.6134	42.0907	42.5680	43.0453	43.5226	44.0000	44.5000	45.0000	45.5000	46.0000	46.5000
	Pressure angle	15° 13′	15° 6′	14° 59′	14° 52′	14° 45′	14° 37′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′
	Tooth depth, inches	2.1971	2.1982	2.1986	2.1989	2.1992	2.1996	2.2000	2.2000	2.2000	2.2000	2.2000	2.2000
49	Center distance, inches	41.6347	42.1134	42.5907	43.0681	43.5453	44.0226	44.5000	45.0000	45.5000	46.0000	46.5000	46.5000
	Pressure angle	15° 13′	15° 5′	14° 58′	14° 52′	14° 45′	14° 37′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′
	Tooth depth, inches	2.1971	2.1983	2.1986	2.1989	2.1992	2.1996	2.2000	2.2000	2.2000	2.2000	2.2000	2.2000
50	Center distance, inches	42.1347	42.6135	43.0908	43.5681	44.0451	44.5226	45.0000	45.5000	46.0000	46.5000	47.0000	47.5000
	Pressure angle	15° 12′	15° 5′	14° 58′	14° 51′	14° 44′	14° 37′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′
	Tooth depth, inches	2.1971	2.1983	2.1986	2.1989	2.1992	2.1996	2.2000	2.2000	2.2000	2.2000	2.2000	2.2000
52	Center distance, inches	43.1349	43.6137	44.0909	44.5682	45.0451	45.5226	46.0000	46.5000	47.0000	47.5000	48.0000	48.5000
	Pressure angle	15° 11′	15° 5′	14° 58′	14° 51′	14° 44′	14° 37′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′
	Tooth depth, inches	2.1973	2.1984	2.1987	2.1990	2.1993	2.1996	2.2000	2.2000	2.2000	2.2000	2.2000	2.2000
54	Center distance, inches	44.1351	44.6139	45.0910	45.5682	46.0454	46.5226	47.0000	47.5000	48.0000	48.5000	49.0000	49.5000
	Pressure angle	15° 11′	15° 4′	14° 57′	14° 50′	14° 44′	14° 37′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′
	Tooth depth, inches	2.1975	2.1985	2.1988	2.1991	2.1993	2.1996	2.2000	2.2000	2.2000	2.2000	2.2000	2.2000
56	Center distance, inches	45.1353	45.6140	46.0911	46.5683	47.0454	47.5226	48.0000	48.5000	49.0000	49.5000	50.0000	50.5000
	Pressure angle	15° 10′	15° 4′	14° 57′	14° 50′	14° 43′	14′ 37°	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′
	Tooth depth, inches	2.1977	2.1986	2.1989	2.1991	2.1993	2.1996	2.2000	2.2000	2.2000	2.2000	2.2000	2.2000
58	Center distance, inches	46.1355	46.6140	47.0912	47.5683	48.0455	48.5226	49.0000	49.5000	50.0000	50.5000	51.0000	51.5000
	Pressure angle	15° 10′	15° 3′	14° 56′	14° 49′	14° 43′	14° 37′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′
	Tooth depth, inches	2.1979	2.1987	2.1990	2.1992	2.1994	2.1996	2.2000	2.2000	2.2000	2.2000	2.2000	2.2000
60	Center distance, inches	47.1356	47.6140	48.0912	48.5684	49.0455	49.5226	50.0000	50.5000	51.0000	51.5000	52.0000	52.5000
	Pressure angle	15° 9′	15° 3′	14° 56′	14° 49′	14° 42′	14° 36′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′	14° 30′
	Tooth depth, inches	2.1980	2.1988	2.1990	2.1992	2.1994	2.1996	2.2000	2.2000	2.2000	2.2000	2.2000	2.2000

to undercut limit, tooth thickness at the 14½-deg. pitch line, that are useful in different gear computations.

The various equations in previous chapters enable the pressure angles and the center distances for any combination of these gears to be computed. The tooth depths would be established, as shown by author. Table XII gives the center distances, pressure angles, and tooth depths for pinions of 10 to 45 teeth meshing with other gears or pinions of from 10 to 60 teeth. It will be noted that for pinions of 40 teeth and larger, the tooth depth is constant and the center distances are proportional to the numbers of teeth. These tables are based on a basic rack of 1 d.p. For other diametral pitches the values given should be divided by the pitch used.

The two foregoing tables enable the dimensions for any pair or train of meshing gears to be determined by simply adding the proper tooth depth to the root radii of the selected combination of gears to obtain the outside radii, while all other dimensions are taken directly from the tables.

As an example, we will take a 20-tooth pinion that is to mesh with a 35-tooth gear. We get, from Table XI,

Number of teeth	20	35
Root radius	9.2577 in.	16.4144 in.

and from Table XII,

Center distance	28.0073 in.
Pressure angle	18 deg. 5 min.
Tooth depth	2.1406 in.

whence, outside radius of pinion

$$= 9.2577 + 2.1406 = 11.3983 \text{ in.}$$

Outside radius of gear

$$= 16.4144 + 2.1406 = 18.5550 \text{ in.}$$

A study of Table XII will show that all combinations of gears with tooth numbers between 10 and 40, whose sum of tooth numbers is the same, will have the same center distance, pressure angle, and tooth depth. For example, the 10-40 combination, the 11-39 combination, the 12-38 combination, etc., all have a center distance of 25.5931 in., a pressure angle of 18 deg. 58 min., and a tooth depth of 2.1143 in. Thus, gear sets of several differ-

ent ratios, the total number of teeth in each pair being the same, will run on a fixed pair of shafts, such as are used in change-gear boxes. They can also be used as change gears, where the centers are adjusted to suit the particular gears selected. Although the smaller gears of this series do not operate on proportional-center distances, they are all interchangeable with each other and can be used as extensively as the conventional systems, provided the center distances are established to suit this series.

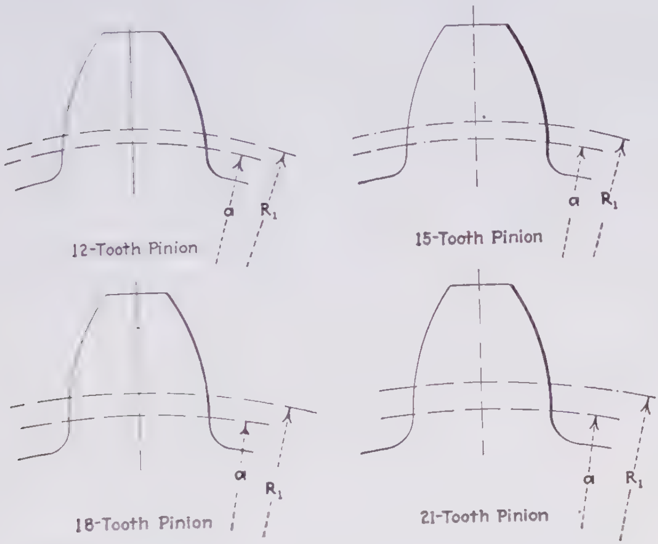


FIG. 73.—Strong tooth forms of the 14½-deg., basic-rack, variable-center system.

We will make a brief analysis of the variable-center system, for the purpose of comparing it with the conventional systems. The forms of the 12-, 15-, 18- and 21-tooth gears of this system are shown in Fig. 73. Particular attention is drawn to the strong tooth forms that result from the avoidance of undercutting.

Table XIII shows the duration of contact, in terms of tooth intervals, between the combinations of gears in this system with small tooth numbers.

As a direct comparison with gears of other systems, we will examine the sliding conditions between various combinations in this series. As a first example, we will consider a pair of 12-tooth pinions of 1 d.p. meshing together. For these, we get the follow-

TABLE XIII.—DURATION OF CONTACT BETWEEN GEARS OF THE 14½-DEG. BASIC-RACK, VARIABLE-CENTER SYSTEM IN TERMS OF TOOTH INTERVALS

Number of teeth in gear	Number of teeth in pinion															
	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
10	0.961															
11	0.981	1.010														
12	1.000	1.030	1.052													
13	1.019	1.049	1.072	1.097												
14	1.038	1.068	1.092	1.117	1.142											
15	1.056	1.087	1.112	1.137	1.162	1.184										
16	1.074	1.106	1.132	1.157	1.181	1.204	1.224									
17	1.092	1.125	1.151	1.176	1.200	1.223	1.244	1.266								
18	1.109	1.143	1.170	1.195	1.218	1.242	1.264	1.286	1.308							
19	1.126	1.160	1.188	1.213	1.236	1.261	1.283	1.305	1.327	1.347						
20	1.142	1.176	1.205	1.230	1.254	1.279	1.302	1.324	1.345	1.365	1.385					
21	1.156	1.191	1.220	1.246	1.271	1.296	1.320	1.341	1.362	1.383	1.404	1.424				
22	1.169	1.205	1.234	1.261	1.287	1.312	1.337	1.358	1.379	1.401	1.423	1.443	1.459			
23	1.181	1.218	1.247	1.275	1.302	1.327	1.353	1.374	1.395	1.418	1.441	1.461	1.480	1.498		
24	1.193	1.231	1.260	1.289	1.317	1.342	1.368	1.389	1.410	1.434	1.458	1.478	1.497	1.516	1.534	
25	1.204	1.243	1.272	1.302	1.331	1.356	1.382	1.403	1.424	1.449	1.474	1.494	1.513	1.532	1.550	1.567
26	1.214	1.254	1.283	1.314	1.344	1.369	1.395	1.416	1.437	1.463	1.489	1.510	1.530	1.550	1.569	1.584
27	1.224	1.264	1.294	1.326	1.356	1.381	1.407	1.428	1.449	1.476	1.503	1.525	1.546	1.567	1.587	1.600
28	1.233	1.273	1.304	1.337	1.367	1.393	1.419	1.440	1.460	1.488	1.516	1.539	1.560	1.581	1.601	1.615
29	1.243	1.283	1.315	1.348	1.378	1.405	1.431	1.451	1.471	1.500	1.528	1.551	1.572	1.593	1.613	1.630
30	1.252	1.292	1.325	1.358	1.389	1.417	1.442	1.462	1.482	1.511	1.540	1.563	1.584	1.605	1.625	1.644
31	1.261	1.301	1.335	1.368	1.399	1.428	1.453	1.474	1.495	1.524	1.553	1.576	1.598	1.620	1.640	1.660
32	1.270	1.310	1.344	1.378	1.409	1.439	1.464	1.486	1.508	1.537	1.566	1.589	1.612	1.635	1.655	1.674
33	1.279	1.319	1.353	1.387	1.419	1.449	1.475	1.498	1.521	1.550	1.579	1.602	1.626	1.649	1.670	1.688
34	1.288	1.328	1.362	1.396	1.429	1.459	1.488	1.510	1.534	1.563	1.592	1.615	1.639	1.663	1.684	1.702
35	1.297	1.337	1.371	1.405	1.438	1.469	1.499	1.522	1.546	1.575	1.604	1.627	1.651	1.676	1.697	1.716
36	1.306	1.346	1.380	1.414	1.447	1.479	1.509	1.534	1.558	1.587	1.616	1.639	1.663	1.688	1.710	1.730
37	1.315	1.355	1.389	1.423	1.456	1.489	1.519	1.545	1.570	1.598	1.628	1.652	1.674	1.700	1.723	1.734
38	1.323	1.364	1.398	1.432	1.465	1.499	1.529	1.556	1.582	1.610	1.640	1.664	1.688	1.712	1.735	1.757
39	1.331	1.373	1.407	1.441	1.474	1.509	1.539	1.567	1.594	1.622	1.651	1.676	1.700	1.724	1.747	1.769
40	1.339	1.381	1.416	1.450	1.483	1.519	1.549	1.578	1.606	1.634	1.662	1.688	1.712	1.736	1.759	1.781

ing values, which are tabulated with the values for similar pinions of the 20-deg. stub-tooth system:

	20-deg. stub-tooth system	14½-deg. basic-rack, variable-center system
Number of teeth	12	12
Maximum radius of curvature of active profile, inches	3.8014	4.4122
Minimum radius of curvature of active profile, inches	0.3028	1.2073
Active profile above pitch line, inches	0.8000	0.7976
Active profile below pitch line, inches	0.3538	0.5190
Total height of active profile, inches	1.1538	1.3166
Specific sliding, addendum	+ 0.92	+0.72
Specific sliding, dedendum	−11.55	−2.65
Duration of contact, tooth intervals	1.185	1.052
Pressure angle	20 deg.	25 deg. 49 min.

The specific sliding on the variable-center gears is plotted in Fig. 74. This table of comparison shows a shortened duration of contact for the variable-center gears but larger radii of curvature, longer active profiles, and much more favorable sliding conditions than on the 20-deg. stub-tooth gears. The additional contact on the stub-tooth gears has been gained at the expense of very much more sensitive tooth profiles. Of the two, much less difficulty

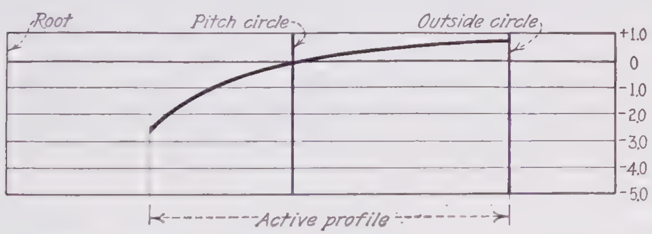


Fig. 74.—Specific sliding between two 12-tooth pinions of the 14½-deg., variable-center system.

would be incurred in producing satisfactory gears to the variable-center system than to the stub-tooth system. Neither of these two pairs is suitable for the quiet transmission of any amount of power, because the contact is too short in both cases.

As a second example, we will compare these same 12-tooth pinions running with 30-tooth gears, 1 d.p. For these, we have the following values:

	20-deg. stub-tooth system	14½-deg. basic-rack, variable-center system
Number of teeth in pinion	12	12
Maximum radius of curvature of active profile, inches	3.8014	4.7416
Minimum radius of curvature of active profile, inches	0.1682	0.7086
Active profile above pitch line, inches	0.8000	1.2943
Active profile below pitch line, inches	0.3593	0.3522
Total height of active profile, inches	1.1593	1.6465
Specific sliding, addendum	+ 0.64	+0.75
Specific sliding, dedendum	−15.68	−2.90
Duration of contact, tooth intervals	1.230	1.325
Pressure angle	20 deg.	20 deg. 34 min.
Number of teeth in gear	30	30
Maximum radius of curvature of active profile, inches	7.0142	6.9196
Minimum radius of curvature of active profile, inches	3.3810	2.8866
Active profile above pitch line, inches	0.7441	0.5763
Active profile below pitch line, inches	0.5048	0.7039
Total height of active profile, inches	1.2489	1.2802
Specific sliding, addendum	+0.94	+0.74
Specific sliding, dedendum	−1.81	−3.10

The specific sliding on the variable-center gears is plotted in Fig. 75. This table of comparison shows larger radii of curvature

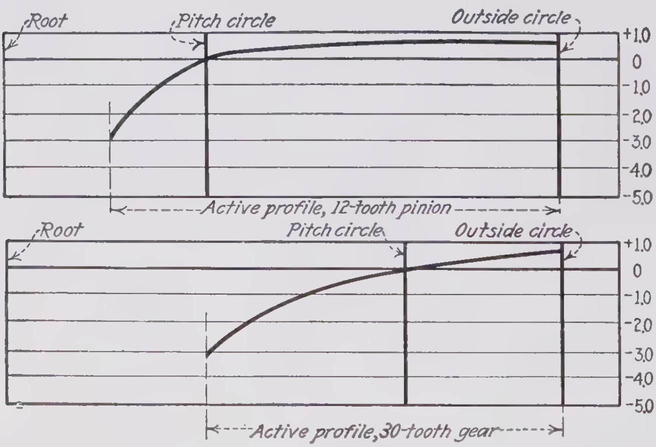


Fig. 75.—Specific sliding between a 12-tooth pinion and a 30-tooth gear, 14½-deg., variable-center system.

for the pinion, longer active profiles, nearly balanced and more favorable sliding conditions, and longer contact for the variable-center gears. The pressure angles are practically the same on

both pairs. This great difference in the nature of the tooth action between them is due to the fact that the tooth design of one pair is governed by the base circles, while that of the conventional pair is governed by the pitch circles.

The next example will be a pair of 14-tooth, 1-d.p. pinions that will be compared with similar pinions of the 20-deg., full-depth tooth system. For these, we have the following values:

	20-deg. full-depth tooth system	14½-deg. basic-rack variable-center system
Number of teeth	14	14
Maximum radius of curvature of active profile, inches	4.4828	4.8024
Minimum radius of curvature of active profile, inches	0.3054	1.3263
Active profile above pitch line, inches	0.9614	0.8685
Active profile below pitch line, inches	0.4150	0.5314
Total height of active profile, inches	1.3764	1.3999
Specific sliding, addendum	+ 0.93	+0.72
Specific sliding, dedendum	−13.67	−2.62
Duration of contact, tooth intervals	1.415	1.142
Pressure angle	20 deg.	24 deg. 20 min.

The specific sliding on the varible-center gears is plotted in Fig. 76 The contact on the 20-deg. full-depth tooth gears is very much greater than on the variable-center gears, but the

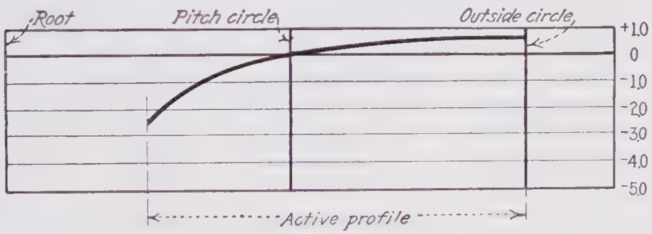


FIG. 76.—Specific sliding between two 14-tooth pinions of the 14½-deg., variable-center system.

additional contact is obtained at the expense of very sensitive and troublesome tooth profiles.

The next example will compare these same 14-tooth pinions when meshing with 30-tooth gears, 1 d.p. For these we have the following values:

	20-deg. full-depth tooth system	14½-deg. basic-rack variable-center system
Number of teeth in pinion	14	14
Maximum radius of curvature of active profile, inches	4.5532	5.0779
Minimum radius of curvature of active profile, inches	0.3054	0.8525
Active profile above pitch line, inches	1.0000	1.2502
Active profile below pitch line, inches	0.4150	0.3878
Total height of active profile, inches	1.4150	1.6380
Specific sliding, addendum	+ 0.69	+0.74
Specific sliding, dedendum	−10.03	−2.41
Duration of contact, tooth intervals	1.347	1.389
Pressure angle	20 deg.	20 deg. 8 min.
Number of teeth in gear	30	30
Maximum radius of curvature of active profile, inches	7.2190	6.9562
Minimum radius of curvature of active profile, inches	2.9712	2.7308
Active profile above pitch line, inches	0.8365	0.5921
Active profile below pitch line, inches	0.5949	0.6909
Total height of active profile, inches	1.4314	1.2830
Specific sliding, addendum	+0.91	+0.73
Specific sliding, dedendum	−2.28	−2.98

The specific sliding on the variable-center gears is plotted in Fig. 77. Here, again, the sliding conditions are balanced and

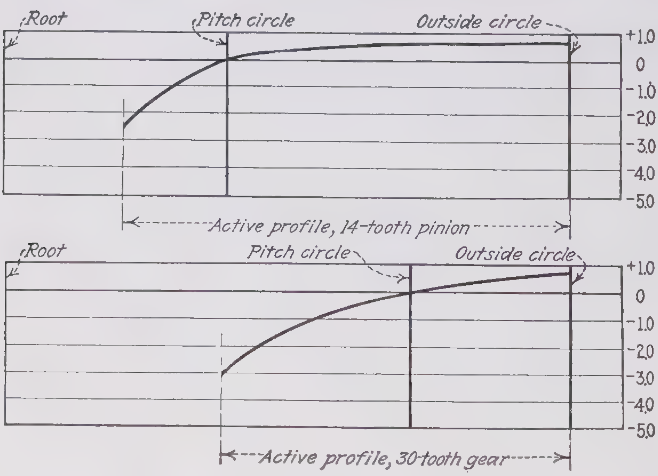


Fig. 77.—Specific sliding between a 14-tooth pinion and a 30-tooth gear, 14½-deg., variable-center system.

much more favorable on the variable-center gears than on the others.

The next example will be a pair of 22-tooth, 1-d.p. gears meshing together. These will be compared with all three of the

conventional tooth systems now in general use. For these, we have the following:

	14½-deg. generated system	20-deg. full-depth tooth system	20-deg. stub-tooth system	14½-deg. basic-rack, variable-center system
Number of teeth	22	22	22	22
Active profile, inches:				
Maximum radius curvature	4.7921	6.0954	5.6916	6.1244
Minimum radius curvature	0.7163	1.4290	1.7328	1.6843
Above pitch line	0.6610	1.0000	0.8000	0.9422
Below pitch line	0.3263	0.5651	0.5192	0.5609
Total height	0.9873	1.5651	1.3192	1.5031
Specific sliding:				
Addendum	+0.85	+0.76	+0.69	+0.72
Dedendum	−5.69	−3.26	−2.28	−2.63
Contact	1.340	1.580	1.307	1.462
Pressure angle	14½ deg.	20 deg.	20 deg.	20 deg. 8 min.

The specific sliding on the variable-center gears is plotted in Fig. 78. It will be noted that these gears are just between the 20-deg. full-depth tooth and the 20-deg. stub-tooth gears.

These variable-center gears, in general, show more favorable and balanced operating and production conditions than those of the conventional systems. With small tooth numbers, it is necessary to use higher pressure angles in order to obtain suffi-

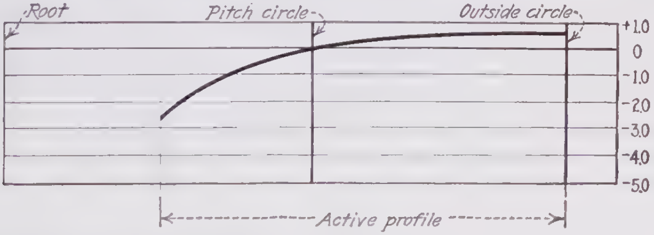


FIG. 78.—Specific sliding between two 22-tooth pinions of the 14½-deg. variable-center system.

cient distance between the base circles to form effective gear-tooth profiles. It is also necessary to vary the height of the addenda of mating gears of small tooth numbers to avoid sensitive and troublesome tooth profiles. As the numbers of teeth increase, the distance between the base circles becomes greater, so that lower pressure angles can be used effectively. When sufficient distance between the base circle exists, it is not essential to vary the addenda of mating gears, because the portion of the

involute curve used for the tooth profiles has flattened out to such an extent that the possibility of obtaining sensitive profiles has disappeared. A system of variable-center gears for the smaller tooth numbers, such as has been described, where the teeth on each gear of the system have been designed to keep away from the base circle and sensitive parts of the involute curve, automatically obtains favorable contact conditions. It will be noted that the specific sliding on these variable-center gears remains nearly constant for all combinations. It will also be found that the strength of the tooth forms is nearly constant throughout the whole range, the smaller tooth numbers being slightly stronger, if anything, than the larger tooth numbers. In other words, such a method of designing tooth forms results in a balanced design.

Although these variable-center gears can be readily used on all new construction, they cannot be used for replacement in existing mechanisms that are made to use gears designed for proportional-center distances. A series of cutters could be made, however, so that these same general tooth designs could be used on such proportional-center distances. There are several possible solutions to this problem. We will consider only the simplest.

Range-cutter, Proportional-center Distance System.—In the variable-center system, the only restriction placed upon the tooth design was that all gears of the same nominal pitch must be generated from a single basic rack. We will now remove that restriction and impose a different one, to wit, that all gears of a given nominal pitch must operate at center distances that are directly proportional to the total number of teeth in the mating pair. In order to accomplish this and still retain effective tooth design, we must base this system on a series of basic racks of different pressure angles and tooth proportions instead of a single one. Such a system will be called the "range-cutter, proportional-center system." For gears of 1 d.p. the center distances will always be equal to one-half the sum of the tooth numbers of the meshing pair. As before, we will study only gears of 1 d.p. The dimensions of gears of other pitches are determined by dividing the values for 1 d.p. by the diametral pitch employed.

Instead of having one basic rack for the gear system and varying the center distances to obtain more effective gear-tooth design, we will now have a series of different basic racks and will

maintain proportional-, or standard-, center distances. A study of the table of center distances and pressure angles for the variable-center distance gear system given in the preceding article shows pressure angles ranging from 14½ up to about 27½ deg. This variation must be divided into steps that will establish the pressure angles for the series of range cutters of basic-rack form.

At the present time, generating cutters, or hobs, of involute basic-rack form with pressure angles of 14½ and 20 deg. are extensively used. We will therefore retain both of these cutters in our proposed series of range cutters. The resulting system of gears will thus be an extension and improvement of existing systems, or, rather, the coordination of the present systems into a single one, instead of an entirely new gear system.

The difference between 14½ and 20 deg. is too large a step, so that we will introduce an intermediate pressure angle of 17 deg. We will also make the greatest pressure angle 25 deg. and introduce an intermediate one of 22½ deg. This gives us the following series of pressure angles: 14½, 17, 20, 22½, and 25 deg.; a total of five. Of these, 14½ and 20 deg. are now extensively used; 17 and 22½ deg. are used occasionally; while only the 25-deg. basic rack is entirely new. As a matter of fact, this last one would be used but seldom, because it is needed only on the smallest pairs.

A further study of Table XII, of pressure angles and tooth depths of the variable-center distance gears, will show that with the higher pressure angles the tooth depths are less. In other words, when the tooth numbers are small, not only is the pressure angle increased in order to obtain a greater distance between the base circles but the tooth depths are also reduced, so that a sufficient distance between these base circles is obtained with a smaller increase in pressure angle than would otherwise be necessary. We will therefore make the basic racks of this series of the higher pressure angles with a lesser working depth. For the sake of simplicity, we will use a constant clearance of 0.200 in. The proportions of these basic racks will therefore be as follows:

Pressure angle, degrees	25	22½	20	17	14½
Addendum, inches	0.800	0.900	1.000	1.000	1.000
Working depth, inches	1.600	1.800	2.000	2.000	2.000
Clearance, inches	0.200	0.200	0.200	0.200	0.200
Whole depth, inches	1.800	2.000	2.200	2.200	2.200

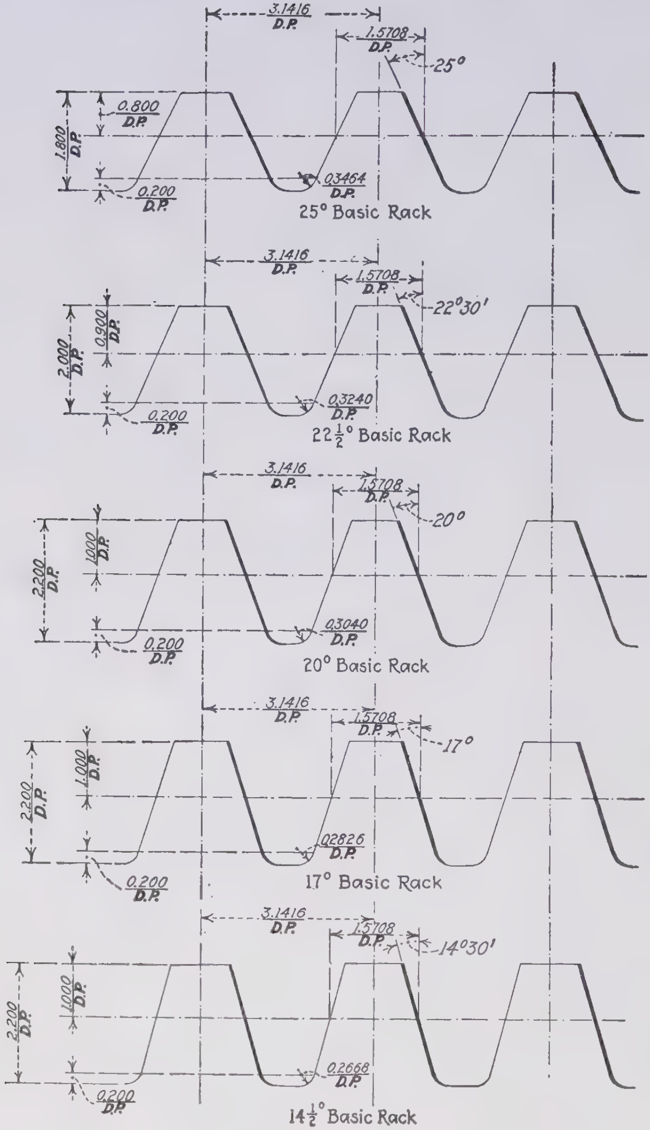


FIG. 79.—Basic racks of the range-cutter, proportional-center system.

These basic racks are shown in Fig. 79. The nominal pitch line of these racks, or the place where the thickness of tooth and space are equal, will be in the middle of the working depth. At this point, the thickness of both the tooth and the space of the basic rack of 1 d.p. is equal to 1.5708 in.

The next step is to establish the range that each basic rack will cover. We will use the 14½-deg. basic rack for all gears where the smallest one in the pair has 40 teeth or more. Referring to Table XII, of pressure angles for the variable-center distance gears, we will establish the following ranges for each of our basic racks:

14½-deg. basic-rack cutter to generate all gears of 14½-deg. pressure angle.
17-deg. basic-rack cutter to generate gears from 14-deg. 31-min. to 17-deg. 20-min. pressure angle.
20-deg. basic-rack cutter to generate gears from 17-deg. 21-min. to 20-deg. 45-min. pressure angle.
22½-deg. basic-rack cutter to generate gears from 20-deg. 46-min. to 23-deg. 15-min. pressure angle.
25-deg. basic-rack cutter to generate all gears over 23-deg. 16-min. pressure angle.

The next step is to determine the proportions of the addenda of mating gears. When the tooth numbers are equal, the addenda must also be equal. In all other cases, however, with pinions of less than 40 teeth, the addendum of the smaller gear, or pinion, should be longer than that of its mating gear, in order to avoid sensitive tooth profiles and other unfavcrable contact conditions. For example, the 10-tooth pinion of the variable-center gears when meshing with a 40-tooth gear has an addendum of 1.4823 in., while that of the mating gear is 0.4399 in. In this case, the addendum of the pinion is over three times as long as that of its mating gear. We will establish the maximum addendum for any pinion as three times that of its mating gear. The sum of the addenda of mating gears is always equal to the working depth, whence, the maximum addendum of any pinion will be three-fourths of the working depth.

The next step is to establish the increments or differences in the addenda for successive combinations. On the variable-center distance gears, this increment starts at about 0.040 in. between the 10-10 and 10-11 combinations, and reduces to about 0.013 in. between the 10-39 and 10-40 combinations. For the sake of simplicity, we will establish a constant increment of 0.020 in., until the pinion meshes with a 40-tooth gear or until the adden-

dum of the pinion reaches a maximum of three-quarters of the working depth, after which the addendum of any pinion will remain constant.

The maximum correction of the addendum of any pinion will be reached when it meshes with a 40-tooth gear. When it meshes with larger gears, it will have the same correction, and we will also use the same pressure angle, as when it meshes with a 40-tooth gear of this system.

A table has been prepared that gives the outside radius of the pinion, the pressure angle of the basic-rack cutter to be employed, and the outside radius of the gear for combinations of pinions of from 10 to 45 teeth meshing with gears of from 10 to 45 teeth, 1 d.p., of the range-cutter, proportional-center distance system. It is not necessary to extend Table XIV further, because the conditions remain constant beyond it.

The center distance for any pair of gears is equal to one-half the sum of the number of teeth in both gears. The tooth depth is constant for any given pressure angle. The tooth depths for each pressure angle are tabulated at the bottom of the table. Dimensions for any pair can be readily determined. As an example, we will take a 20-tooth pinion that is to mesh with a 35-tooth gear. From the table, we get

Number of teeth	20	35
Outside radius	11.300 in.	18.200 in.
Pressure angle	20 deg.	
Tooth depth	2.200 in.	

Whence,

Root radius	9.100 in.	16.000 in.

We also know that the center distance is equal to one-half of the sum of the tooth numbers, whence,

Center distance.............................. 27.500 in.

These dimensions are for gears of 1 d.p. For gears of any other pitch, these figures would be divided by the diametral pitch desired.

As a second example, we will take a set of gears that is beyond the tabulated pairs, that is, a 12-tooth pinion that meshes with a 72-tooth gear. We find, from the table, that the maximum outside radius of the 12-tooth pinion is 7.500 in. and that the pressure angle is 20 deg. The addendum of this pinion has been *increased* 0.500 in., and, hence, the addendum of its mating gear must be *decreased* the same amount, with the result that the

outside radius of the 72-tooth gear will be 36.500 in. Thus, we have, for this pair,

Number of teeth	12	72
Outside radius	7.500 in.	36.500 in.
Root radius	5.300 in.	34.300 in.
Pressure angle	20 deg.	
Center distance	42.000 in.	

We will now make a brief analysis of this system to insure that we have retained favorable contact conditions. This analysis should show a fairly close approach to the variable-center distance system, because that system has been used as a guide in establishing the general proportions of the gears in this proportional-center distance system. We will confine this analysis to those pairs that contain the 10-tooth pinion, because the most unfavorable conditions will exist on these 10-tooth pinion drives.

The first example will be a pair of 10-tooth pinions.

N = number of teeth = 10
E = outside radius = 5.800 in.
R = 25-deg. pitch radius = 5.0000 in.
F = addendum of basic-rack form cutter, including clearance, = 1.0000 in.
f = clearance = 0.2000 in.
H = root radius = 4.0000 in.
α = pressure angle of cutter and gears = 25 deg.
A = minimum root radius without undercut
a = radius of base circle
Pn = normal pitch

$$a = R\cos\alpha = 4.5316 \text{ in.} \qquad \text{(see Eq. (55))}$$
$$A = R\cos^2\alpha - f = 3.9070 \text{ in.} \qquad \text{(see Eq. (60))}$$

This pinion is not undercut, as the root radius is outside of the undercut limit.

The next step is to determine the thickness of the tooth at the tip, to make sure that the former is not pointed. We have, from Prob. 6, Chap. III,

$$\cos\alpha_2 = \frac{r_1\cos\alpha_1}{r_2} \qquad \text{(see Eq. (39))}$$
$$= 2r_2\left(\frac{T_1}{2r_1} + \operatorname{inv}\alpha_1 - \operatorname{inv}\alpha_2\right) \qquad \text{(see Eq. (40))}$$

In this example,

$$r_1 = 5.0000 \text{ in.}$$
$$r_2 = 5.8000 \text{ in.}$$
$$T_1 = 1.5708 \text{ in.}$$
$$\alpha_1 = 25 \text{ deg.}$$

whence,

$$\cos \alpha_2 = 0.78130 \text{ and } \alpha_2 = 38 \text{ deg. } 37 \text{ min.}$$
$$\text{inv } \alpha_2 = 0.12478$$

whence,

$$T_2 = 0.7224 \text{ in.} = \text{thickness at tip}$$

We will now examine the contact conditions between these two 10-tooth pinions. We have, from Eq. (22), in Chap. II,

$$\text{Number of teeth in contact} = \frac{\sqrt{E_1^2 - a_1^2} + \sqrt{E_2^2 - a_2^2} - C \sin \alpha}{Pn}$$

In this example, both gears are alike.

$$\sqrt{E^2 - a^2} = 3.6201$$
$$C \sin \alpha = 4.2262$$
$$Pn = 2.8473 \text{ in}$$

Whence, contact = 1.058 tooth intervals

This contact is a little greater than on the similar pair of variable-center distance gears.

An examination of the specific sliding conditions shows the following:

Maximum radius of curvature of active profile..	3.6201 in.
Minimum radius of curvature of active profile..	0.6061 in.
Specific sliding, addendum..................	+0.83
Specific sliding, dedendum..................	−4.98

This specific sliding is but slightly greater than on the similar pair of variable-center distance gears. These gears approximate very closely, in all respects, variable-center distance gears.

We will now examine a 10-tooth pinion that meshes with a 21-tooth gear. This is the last gear in this series that employs a 25-deg. cutter. We have, for these gears,

Number of teeth..................	$N_1 = 10$	$N_2 = 21$
Outside radius....................	$E_1 = 6.0200$ in.	$E_2 = 11.0800$ in.
Root radius.......................	$H_1 = 4.2200$ in.	$H_2 = 9.2800$ in.
Base circle radius................	$a_1 = 4.5316$ in.	$a_2 = 9.5163$ in.
Pressure angle....................	$\alpha = 25$ deg.	
Normal pitch......................	$Pn = 2.8473$ in.	
Center distance...................	$C = 15.5000$ in.	

Calculating as before, we get

Thickness at tip of 10-tooth pinion tooth	= 0.6221 in.	
Contact	= 1.080 tooth intervals	
Number of teeth	10	21
Maximum radius of curvature of active profile	3.9619 in.	5.6752 in.
Minimum radius of curvature of active profile	0.7859 in.	2.5887 in.
Specific sliding, addendum	+0.68	+0.67
Specific sliding, dedendum	−2.08	−2.21

The contact on this pair is slightly less than on the same combination of variable center distance gears. The sliding conditions are favorable and well balanced. Hence this pair also approximates a similar pair of variable center distance gears.

We will now examine the conditions that exist when a 10-tooth pinion meshes with a 22-tooth gear. This is the first pair in this series that employs a 22½-deg. cutter. We have, for these gears,

Number of teeth	$N_1 = 10$	$N_2 = 22$
Outside radius	$E_1 =$ 6.1400 in.	$E_2 =$ 11.6600 in.
Pitch radius	$R_1 =$ 5.0000 in.	$R_2 =$ 11.0000 in.
Root radius	$H_1 =$ 4.1400 in.	$H_2 =$ 9.6600 in.
Undercut radius	$A_1 =$ 4.0678 in.	$A_2 =$ 9.3891 in.
Radius of base circle	$a_1 =$ 4.6194 in.	$a_2 =$ 10.1627 in.
Pressure angle	$\alpha =$ 22 deg. 30 min.	
Normal pitch	$Pn =$ 2.9025 in.	
Center distance	$C =$ 16.0000 in.	

Calculating as before, we get:

Thickness at tip of 10-tooth pinion tooth	= 0.5171	
Contact	= 1.253 tooth intervals	
Number of teeth	10	22
Maximum radius of curvature of active profile	4.0448 in.	5.7162 in.
Minimum radius of curvature of active profile	0.4067 in.	2.0781 in
Specific sliding, addendum	+0.76	+0.84
Specific sliding, dedundum	−5.39	−3.28

This pair is a fairly close approximation to the same combination of variable-center distance gears. The contact is greater, but the sliding conditions are not so well balanced.

We will take, as the next example, a 10-tooth pinion that meshes with a 31-tooth gear. This is the last pair in this series that employs a 22½-deg. cutter. For these gears, we have the following values:

Number of teeth	$N_1 = 10$	$N_2 = 31$
Outside radius	$E_1 =$ 6.3200 in.	$E_2 =$ 15.9800 in.
Pitch radius	$R_1 =$ 5.0000 in.	$R_2 =$ 15.5000 in.
Root radius	$H_1 =$ 4.3200 in.	$H_2 =$ 13.9800 in.
Radius of base circle	$a_1 =$ 4.6105 in.	$a_2 =$ 14.3201 in.
Pressure angle	$\alpha =$ 22 deg. 30 min.	
Normal pitch	$Pn =$ 2.9025 in.	
Center distance	$C =$ 20.5000 in.	

Calculating as before, we get

Thickness at tip of 10-tooth pinion tooth	= 0.4995 in.	
Contact	= 1.226 tooth intervals	
Number of teeth	10	31
Maximum radius of curvature of active profile	4.3132 in.	7.0918 in.
Minimum radius of curvature of active profile	0.7531 in.	3.5317 in.
Specific sliding, addendum	+0.73	+0.67
Specific sliding, dedendum	−2.04	−2.78

This pair is also a very close approximation to the same combination of variable center distance gears. The contact is slightly less, but the sliding conditions are favorable and very well balanced.

We will now examine the conditions that exist when a 10-tooth pinion meshes with a 32-tooth gear, since this is the first pair in this series that employs a 20-deg. cutter. We have, for these gears,

Number of teeth	$N_1 = 10$	$N_2 = 32$
Outside radius	$E_1 =$ 6.4400 in.	$E_2 =$ 16.5600 in.
Pitch radius	$R_1 =$ 5.0000 in.	$R_2 =$ 16.0000 in.
Root radius	$H_1 =$ 4.2400 in.	$H_2 =$ 14.3600 in.
Undercut radius	$A_1 =$ 4.2151 in.	$A_2 =$ 13.9283 in.
Radius of base circle	$a_1 =$ 4.6985 in.	$a_2 =$ 15.0350 in.
Pressure angle	$\alpha =$ 20 deg.	
Normal pitch	$Pn =$ 2.9521 in.	
Center distance	$C =$ 21.0000 in.	

Calculating as before, we get

Thickness of tip of 10-tooth pinion tooth	= 0.2537 in.	
Contact	= 1.410 tooth intervals	
Number of teeth	10	32
Maximum radius of curvature of active profile	4.4043 in.	6.9412 in.
Minimum radius of curvature of active profile	0.2412 in.	2.7781 in.
Specific sliding, addendum	+0.80	+0.88
Specific sliding, dedendum	−7.99	−4.07

The contact on this pair is considerably greater than on the similar pair of variable center distance gears, but the specific sliding is greater and not as well balanced. The additional contact has been obtained at the expense of more sensitive tooth profiles. This pair departs more from the variable center distance gears than any other pair of the series. It is a reasonably close approximation to them, however.

For the last example in this series, we will examine the conditions that exist between a 10-tooth pinion and its mating 40-tooth gear. For these gears, we have the following values:

Number of teeth..................	$N_1 = 10$	$N_2 = 40$
Outside radius....................	$E_1 = 6.5000$ in.	$E_2 = 20.5000$ in.
Pitch radius.......................	$R_1 = 5.0000$ in.	$R_2 = 20.0000$ in.
Root radius........................	$H_1 = 4.3000$ in.	$H_2 = 18.3000$ in.
Radius of base circle..............	$a_1 = 4.6985$ in.	$a_2 = 18.7938$ in.
Pressure angle.....................	$\alpha = 20°$	
Normal pitch.......................	$Pn = 2.9521$ in.	
Center distance....................	$C = 25.0000$ in.	

Calculating as before, we get

Thickness of tip of 10-tooth pinion tooth..........................	= 0.1976 in.	
Contact..........................	= 1.398 tooth intervals	
Number of teeth..................	10	40
Maximum radius of curvature of active profile..................	4.4916 in.	8.1879 in.
Minimum radius of curvature of active profile..................	0.3626 in.	4.0589 in.
Specific sliding, addendum.........	+0.77	+0.82
Specific sliding, dedendum.........	−4.64	−3.42

This pair is a close approximation to the same combination of variable center distance gears. The contact is slightly more and the specific sliding is slightly greater than would be in the latter case, but conditions are reasonably well balanced.

It is not necessary to check any further. The 10-tooth pinion will show the greatest departures from the variable center distance gears of any in the series because it is the smallest of the series, and the smallest gear of any series is always the most sensitive one. This range-cutter, proportional center distance system is a very close approximation in all respects to the variable center distance gears.

TABLE XIV.—OUTSIDE RADII AND PRESSURE ANGLES OF THE RANGE-CUTTER, PROPORTIONAL-CENTER DISTANCE SYSTEM, 1-D.P. GEARS

	Number of teeth in gear	Number of teeth in pinion 10	11	12	13	14	15	16	17	18	19	20	21
E_1 α E_2	10	5.800 25° 5.800											
E_1 α E_2	11	5.820 25° 6.280	6.300 25° 6.300										
E_1 α E_2	12	5.840 25° 6.760	6.320 25° 6.780	6.800 25° 6.800									
E_1 α E_2	13	5.860 25° 7.240	6.340 25° 7.260	6.820 25° 7.280	7.300 25° 7.300								
E_1 α E_2	14	5.880 25° 7.720	6.360 25° 7.740	6.840 25° 7.760	7.320 25° 7.780	7.800 25° 7.800							
E_1 α E_2	15	5.900 25° 8.200	6.380 25° 8.220	6.860 25° 8.240	7.340 25° 8.260	7.820 25° 8.280	8.300 25° 8.300						
E_1 α E_2	16	5.920 25° 8.680	6.400 25° 8.700	6.880 25° 8.720	7.360 25° 8.740	7.840 25° 8.760	8.320 25° 8.780	8.900 22° 30′ 8.900					
E_1 α E_2	17	5.940 25° 9.160	6.420 25° 9.180	6.900 25° 9.200	7.380 25° 9.220	7.860 25° 9.240	8.440 22° 30′ 9.360	8.920 22° 30′ 9.380	9.400 22° 30′ 9.400				
E_1 α E_2	18	5.960 25° 9.640	6.440 25° 9.660	6.920 25° 9.680	7.400 25° 9.700	7.980 22° 30′ 9.820	8.460 22° 30′ 9.840	8.940 22° 30′ 9.860	9.420 22° 30′ 9.880	9.900 22° 30′ 9.900			

E_1 α E_2	19	5.980 25° 10.120	6.460 25° 10.140	6.940 25° 10.160	7.520 22° 30′ 10.280	8.000 22° 30′ 10.300	8.480 22° 30′ 10.320	8.960 22° 30′ 10.340	9.440 22° 30′ 10.360	9.920 22° 30′ 10.380	10.400 22° 30′ 10.400		
E_1 α E_2	20	6.000 25° 10.600	6.480 25° 10.620	7.060 22° 30′ 10.740	7.540 22° 30′ 10.760	8.020 22° 30′ 10.780	8.500 22° 30′ 10.800	8.980 22° 30′ 10.820	9.460 22° 30′ 10.840	9.940 22° 30′ 10.860	10.420 22° 30′ 10.880	10.900 22° 30′ 10.900	
E_1 α E_2	21	6.020 25° 11.080	6.600 22° 30′ 11.200	7.080 22° 30′ 11.220	7.560 22° 30′ 11.240	8.040 22° 30′ 11.260	8.520 22° 30′ 11.280	9.000 22° 30′ 11.300	9.480 22° 30′ 11.320	9.960 22° 30′ 11.340	10.440 22° 30′ 11.360	10.920 22° 30′ 11.380	11.500 20° 11.500
E_1 α E_2	22	6.140 22° 30′ 11.660	6.620 22° 30′ 11.680	7.100 22° 30′ 11.700	7.580 22° 30′ 11.720	8.060 22° 30′ 11.740	8.540 22° 30′ 11.760	9.020 22° 30′ 11.780	9.500 22° 30′ 11.800	9.980 22° 30′ 11.820	10.460 22° 30′ 11.840	11.040 20° 11.960	11.520 20° 11.980
E_1 α E_2	23	6.160 22° 30′ 12.140	6.640 22° 30′ 12.160	7.120 22° 30′ 12.180	7.600 22° 30′ 12.200	8.080 22° 30′ 12.220	8.560 22° 30′ 12.240	9.040 22° 30′ 12.260	9.520 22° 30′ 12.280	10.000 22° 30′ 12.300	10.580 20° 12.420	11.060 20° 12.440	11.540 20° 12.460
E_1 α E_2	24	6.180 22° 30′ 12.620	6.660 22° 30′ 12.640	7.140 22° 30′ 12.660	7.620 22° 30′ 12.680	8.100 22° 30′ 12.700	8.580 22° 30′ 12.720	9.060 22° 30′ 12.740	9.540 22° 30′ 12.760	10.120 20° 12.880	10.600 20° 12.900	11.080 20° 12.920	11.560 20° 12.940
E_1 α E_2	25	6.200 22° 30′ 13.100	6.680 22° 30′ 13.120	7.160 22° 30′ 13.140	7.640 22° 30′ 13.160	8.120 22° 30′ 13.180	8.600 22° 30′ 13.200	9.080 22° 30′ 13.220	9.660 20° 13.340	10.140 20° 13.360	10.620 20° 13.380	11.100 20° 13.400	11.580 20° 13.420
E_1 α E_2	26	6.220 22° 30′ 13.580	6.700 22° 30′ 13.600	7.180 22° 30′ 13.620	7.660 22° 30′ 13.640	8.140 22° 30′ 13.660	8.620 22° 30′ 13.680	9.200 20° 13.800	9.680 20° 13.820	10.160 20° 13.840	10.640 20° 13.860	11.120 20° 13.880	11.600 20° 13.900
E_1 α E_2	27	6.240 22° 30′ 14.060	6.720 22° 30′ 14.080	7.200 22° 30′ 14.100	7.680 22° 30′ 14.120	8.160 22° 30′ 14.140	8.740 20° 14.260	9.220 20° 14.280	9.700 20° 14.300	10.180 20° 14.320	10.660 20° 14.340	11.140 20° 14.360	11.620 20° 14.380
E_1 α E_2	28	6.260 22° 30′ 14.540	6.740 22° 30′ 14.560	7.220 22° 30′ 14.580	7.700 22° 30′ 14.600	8.280 20° 14.720	8.760 20° 14.740	9.240 20° 14.760	9.720 20° 14.780	10.200 20° 14.800	10.680 20° 14.820	11.160 20° 14.840	11.640 20° 14.860

Legend: E_1 = outside radius of pinion, inches. α = pressure angle of basic rack. E_2 = outside radius of gear, inches.

The working depths vary with the pressure angles as follows:

Pressure angle:	25 deg.	22 deg. 30 min.	20 deg.	17 deg.	14 deg. 30 min.
Working depth:	1.800 in.	2.000 in.	2.200 in.	2.200 in.	2.200 in.

TABLE XIV.—OUTSIDE RADII AND PRESSURE ANGLES OF THE RANGE-CUTTER, PROPORTIONAL-CENTER DISTANCE SYSTEM, 1-D.P. GEARS (*Continued*)

	Number of teeth in gear	Number of teeth in pinion											
		10	11	12	13	14	15	16	17	18	19	20	21
E_1 $\propto$ E_2	29	6.280 22° 30′ 15.020	6.760 22° 30′ 15.040	7.240 22° 30′ 15.060	7.820 20° 15.180	8.300 20° 15.200	8.780 20° 15.220	9.260 20° 15.240	9.740 20° 15.260	10.220 20° 15.280	10.700 20° 15.300	11.180 20° 15.320	11.660 20° 15.340
E_1 $\propto$ E_2	30	6.300 22° 30′ 15.500	6.780 22° 30′ 15.520	7.360 20° 15.640	7.840 20° 15.660	8.320 20° 15.680	8.800 20° 15.700	9.280 20° 15.720	9.760 20° 15.740	10.240 20° 15.760	10.720 20° 15.780	11.200 20° 15.800	11.680 20° 15.820
E_1 E_2	31	6.320 22° 30′ 15.980	6.900 20° 16.100	7.380 20° 16.120	7.860 20° 16.140	8.340 20° 16.160	8.820 20° 16.180	9.300 20° 16.200	9.780 20° 16.220	10.260 20° 16.240	10.740 20° 16.260	11.220 20° 16.280	11.700 20° 16.300
E_1 $\propto$ E_2	32	6.440 20° 16.560	6.920 20° 16.580	7.400 20° 16.600	7.880 20° 16.620	8.360 20° 16.640	8.840 20° 16.660	9.320 20° 16.680	9.800 20° 16.700	10.280 20° 16.720	10.760 20° 16.740	11.240 20° 16.760	11.720 20° 16.780
E_1 $\propto$ E_2	33	6.460 20° 17.040	6.940 20° 17.060	7.420 20° 17.080	7.900 20° 17.100	8.380 20° 17.120	8.860 20° 17.140	9.340 20° 17.160	9.820 20° 17.180	10.300 20° 17.200	10.780 20° 17.220	11.260 20° 17.240	11.740 20° 17.260
E_1 $\propto$ E_2	34	6.480 20° 17.520	6.960 20° 17.540	7.440 20° 17.560	7.920 20° 17.580	8.400 20° 17.600	8.880 20° 17.620	9.360 20° 17.640	9.840 20° 17.660	10.320 20° 17.680	10.800 20° 17.700	11.280 20° 17.720	11.760 20° 17.740
E_1 $\propto$ E_2	35	6.500 20° 18.000	6.980 20° 18.020	7.460 20° 18.040	7.940 20° 18.060	8.420 20° 18.080	8.900 20° 18.100	9.380 20° 18.120	9.860 20° 18.140	10.340 20° 18.160	10.820 20° 18.180	11.300 20° 18.200	11.780 20° 18.220
E_1 $\propto$ E_2	36	6.500 20° 18.500	7.000 20° 18.500	7.480 20° 18.520	7.960 20° 18.540	8.440 20° 18.560	8.920 20° 18.580	9.400 20° 18.600	9.880 20° 18.620	10.360 20° 18.640	10.840 20° 18.660	11.320 20° 18.680	11.800 20° 18.700
E_1 $\propto$ E_2	37	6.500 20° 19.000	7.000 20° 19.000	7.500 20° 19.000	7.980 20° 19.020	8.460 20° 19.040	8.940 20° 19.060	9.420 20° 19.080	9.900 20° 19.100	10.380 20° 19.120	10.860 20° 19.140	11.340 20° 19.160	11.820 20° 19.180

E_1 α E_2	38	6.500 20° 19.500	7.000 20° 19.500	7.500 20° 19.500	8.000 20° 19.500	8.480 20° 19.520	8.960 20° 19.540	9.440 20° 19.560	9.920 20° 19.580	10.400 20° 19.600	10.880 20° 19.620	11.360 20° 19.640	11.840 20° 19.660
E_1 α E_2	39	6.500 20° 20.000	7.000 20° 20.000	7.500 20° 20.000	8.000 20° 20.000	8.500 20° 20.000	8.980 20° 20.020	9.460 20° 20.040	9.940 20° 20.060	10.420 20° 20.080	10.900 20° 20.100	11.380 20° 20.120	11.860 17° 20.140
E_1 α E_2	40	6.500 20° 20.500	7.000 20° 20.500	7.500 20° 20.500	8.000 20° 20.500	8.500 20° 20.500	9.000 20° 20.500	9.480 20° 20.520	9.960 20° 20.540	10.440 20° 20.560	10.920 20° 20.580	11.400 17° 20.600	11.880 17° 20.620
E_1 α E_2	41	6.500 20° 21.000	7.000 20° 21.000	7.500 20° 21.000	8.000 20° 21.000	8.500 20° 21.000	9.000 20° 21.000	9.480 20° 21.020	9.960 20° 21.040	10.440 20° 21.060	10.920 20° 21.080	11.400 17° 21.100	11.880 17° 21.120
E_1 α E_2	42	6.500 20° 21.500	7.000 20° 21.500	7.500 20° 21.500	8.000 20° 21.500	8.500 20° 21.500	9.000 20° 21.500	9.480 20° 21.520	9.960 20° 21.540	10.440 20° 21.560	10.920 20° 21.580	11.400 17° 21.600	11.880 17° 21.620
E_1 α E_2	43	6.500 20° 22.000	7.000 20° 22.000	7.500 20° 22.000	8.000 20° 22.000	8.500 20° 22.000	9.000 20° 22.000	9.480 20° 22.020	9.960 20° 22.040	10.440 20° 22.060	10.920 20° 22.080	11.400 17° 22.100	11.880 17° 22.120
E_1 α E_2	44	6.500 20° 22.500	7.000 20° 22.500	7.500 20° 22.500	8.000 20° 22.500	8.500 20° 22.500	9.000 20° 22.500	9.480 20° 22.520	9.960 20° 22.540	10.440 20° 22.560	10.920 20° 22.580	11.400 17° 22.600	11.880 17° 22.620
E_1 α E_2	45	6.500 20° 23.000	7.000 20° 23.000	7.500 20° 23.000	8.000 20° 23.000	8.500 20° 23.000	9.000 20° 23.000	9.480 20° 23.020	9.960 20° 23.040	10.440 20° 23.060	10.920 20° 23.080	11.400 17° 23.100	11.880 17° 23.120
		22	23	24	25	26	27	28	29	30	31	32	33
E_1 α E_2	22	12.000 20° 12.000											
E_1 α E_2	23	12.020 20° 12.480	12.500 20° 12.500										

Legend: E_1 = outside radius of pinion, inches. α = pressure angle of basic rack. E_2 = outside radius of gear, inches. The working depths vary with the pressure angles as follows:

Pressure angle:	25 deg.	22 deg. 30 min.	20 deg	17 deg.	14 deg. 30 min.
Working depth:	1.800 in.	2.000 in.	2.200 in.	2.200 in.	2.200 in.

TABLE XIV.—OUTSIDE RADII AND PRESSURE ANGLES OF THE RANGE-CUTTER, PROPORTIONAL-CENTER DISTANCE SYSTEM, 1-D.P. GEARS (*Continued*)

	Number of teeth in gear	Number of teeth in pinion											
		22	23	24	25	26	27	28	29	30	31	32	33
E_1 α E_2	24	12.040 20° 12.960	12.520 20° 12.980	13.000 20° 13.000									
E_1 α E_2	25	12.060 20° 13.440	12.540 20° 13.460	13.020 20° 13.480	13.500 20° 13.500								
E_1 α E_2	26	12.080 20° 13.920	12.560 20° 13.940	13.040 20° 13.960	13.520 20° 13.980	14.000 20° 14.000							
E_1 α E_2	27	12.100 20° 14.400	12.580 20° 14.420	13.060 20° 14.440	13.540 20° 14.460	14.020 20° 14.480	14.500 20° 14.500						
E_1 α E_2	28	12.120 20° 14.880	12.600 20° 14.900	13.080 20° 14.920	13.560 20° 14.940	14.040 20° 14.960	14.520 20° 14.980	15.000 20° 15.000					
E_1 α E_2	29	12.140 20° 15.360	12.620 20° 15.380	13.100 20° 15.400	13.580 20° 15.420	14.060 20° 15.440	14.540 20° 15.460	15.020 20° 15.480	15.500 20° 15.500				
E_1 α E_2	30	12.160 20° 15.840	12.640 20° 15.860	13.120 20° 15.880	13.600 20° 15.900	14.080 20° 15.920	14.560 20° 15.940	15.040 20° 15.960	15.520 20° 15.980	16.000 17° 16.000			
E_1 α E_2	31	12.180 20° 16.320	12.660 20° 16.340	13.140 20° 16.360	13.620 20° 16.380	14.100 20° 16.400	14.580 20° 16.420	15.060 20° 16.440	15.540 17° 16.460	16.020 17° 16.480	16.500 17° 16.500		
E_1 α E_2	32	12.200 20° 16.800	12.680 20° 16.820	13.160 20° 16.840	13.640 20° 16.860	14.120 20° 16.880	14.600 20° 16.900	15.080 17° 16.920	15.560 17° 16.940	16.040 17° 16.960	16.520 17° 16.980	17.000 17° 17.000	

E_1 α E_2	33	12.220 20° 17.280	12.700 20° 17.300	13.180 20° 17.320	13.660 20° 17.340	14.140 20° 17.360	14.620 17° 17.380	15.100 17° 17.400	15.580 17° 17.420	16.060 17° 17.440	16.540 17° 17.460	17.020 17° 17.480	17.500 17° 17.500
E_1 α E_2	34	12.240 20° 17.760	12.720 20° 17.780	13.200 20° 17.800	13.680 20° 17.820	14.160 17° 17.840	14.640 17° 17.860	15.120 17° 17.880	15.600 17° 17.900	16.080 17° 17.920	16.560 17° 17.940	17.040 17° 17.960	17.520 17° 17.980
E_1 α E_2	35	12.260 20° 18.240	12.740 20° 18.260	13.220 20° 18.280	13.700 17° 18.300	14.180 17° 18.320	14.660 17° 18.340	15.140 17° 18.360	15.620 17° 18.380	16.100 17° 18.400	16.580 17° 18.420	17.060 17° 18.440	17.540 17° 18.460
E_1 α E_2	36	12.280 20° 18.720	12.760 20° 18.740	13.240 17° 18.760	13.720 17° 18.780	14.200 17° 18.800	14.680 17° 18.820	15.160 17° 18.840	15.640 17° 18.860	16.120 17° 18.880	16.600 17° 18.900	17.080 17° 18.920	17.560 17° 18.940
E_1 α E_2	37	12.300 20° 19.200	12.780 17° 19.220	13.260 17° 19.240	13.740 17° 19.260	14.220 17° 19.280	14.700 17° 19.300	15.180 17° 19.320	15.660 17° 19.340	16.140 17° 19.360	16.620 17° 19.380	17.100 17° 19.400	17.580 17° 19.420
E_1 α E_2	38	12.320 17° 19.680	12.800 17° 19.700	13.280 17° 19.720	13.760 17° 19.740	14.240 17° 19.760	14.720 17° 19.780	15.200 17° 19.800	15.680 17° 19.820	16.160 17° 19.840	16.640 17° 19.860	17.120 17° 19.880	17.600 17° 19.900
E_1 α E_2	39	12.340 17° 20.160	12.820 17° 20.180	13.300 17° 20.200	13.780 17° 20.220	14.260 17° 20.240	14.740 17° 20.260	15.220 17° 20.280	15.700 17° 20.300	16.180 17° 20.320	16.660 17° 20.340	17.140 17° 20.360	17.620 17° 20.380
E_1 α E_2	40	12.360 17° 20.640	12.840 17° 20.660	13.320 17° 20.680	13.800 17° 20.700	14.280 17° 20.720	14.760 17° 20.740	15.240 17° 20.760	15.720 17° 20.780	16.200 17° 20.800	16.680 17° 20.820	17.160 17° 20.840	17.640 17° 20.860
E_1 α E_2	41	12.360 17° 21.140	12.840 17° 21.160	13.320 17° 21.180	13.800 17° 21.200	14.280 17° 21.220	14.760 17° 21.240	15.240 17° 21.260	15.720 17° 21.280	16.200 17° 21.300	16.680 17° 21.320	17.160 17° 21.340	17.640 17° 21.360
E_1 α E_2	42	12.360 17° 21.640	12.840 17° 21.660	13.320 17° 21.680	13.800 17° 21.700	14.280 17° 21.720	14.760 17° 21.740	15.240 17° 21.760	15.720 17° 21.780	16.200 17° 21.800	16.680 17° 21.820	17.160 17° 21.840	17.640 17° 21.860

Legend: E_1 = outside radius of pinion, inches. α = pressure angle of basic rack. E_2 = outside radius of gear, inches. The working depths vary with the pressure angles as follows:

Pressure angle:	25 deg.	22 deg. 30 min.	20 deg.	17 deg.	14 deg. 30 min.
Working depth:	1.800 in.	2.000 in.	2.200 in.	2.200 in.	2.200 in.

TABLE XIV.—OUTSIDE RADII AND PRESSURE ANGLES OF THE RANGE-CUTTER, PROPORTIONAL-CENTER DISTANCE SYSTEM, 1-D.P. GEARS (*Continued*)

	Number of teeth in gear	Number of teeth in pinion											
		22	23	24	25	26	27	28	29	30	31	32	33
E_1 $\propto$ E_2	43	12.360 17° 22.140	12.840 17° 22.160	13.320 17° 22.180	13.800 17° 22.200	14.280 17° 22.220	14.760 17° 22.240	15.240 17° 22.260	15.720 17° 22.280	16.200 17° 22.300	16.680 17° 22.320	17.160 17° 22.340	17.640 17° 22.360
E_1 $\propto$ E_2	44	12.360 17° 22.640	12.840 17° 22.660	13.320 17° 22.680	13.800 17° 22.700	14.280 17° 22.720	14.760 17° 22.740	15.240 17° 22.760	15.720 17° 22.780	16.200 17° 22.800	16.680 17° 22.820	17.160 17° 22.840	17.640 17° 22.860
E_1 $\propto$ E_2	45	12.360 17° 23.140	12.840 17° 23.160	13.320 17° 23.180	13.800 17° 23.200	14.280 17° 23.220	14.760 17° 23.240	15.240 17° 23.260	15.720 17° 23.280	16.200 17° 23.300	16.680 17° 23.320	17.160 17° 23.340	17.640 17° 23.360
		34	35	36	37	38	39	40	41	42	43	44	45
E_1 $\propto$ E_2	34	18.000 17° 18.000											
E_1 $\propto$ E_2	35	18.020 17° 18.480	18.500 17° 18.500										
E_1 $\propto$ E_2	36	18.040 17° 18.960	18.520 17° 18.980	19.000 17° 19.000									
E_1 $\propto$ E_2	37	18.060 17° 19.440	18.540 17° 19.460	19.020 17° 19.480	19.500 17° 19.500								
E_1 $\propto$ E_2	38	18.080 17° 19.920	18.560 17° 19.940	19.040 17° 19.960	19.520 17° 19.980	20.000 17° 20.000							

E_1 α E_2	39	18.100 17° 20.400	18.580 17° 20.420	19.060 17° 20.440	19.540 17° 20.460	20.020 17° 20.480	20.500 17° 20.500						
E_1 α E_2	40	18.120 17° 20.880	18.600 17° 20.900	19.080 17° 20.920	19.560 17° 20.940	20.040 17° 20.960	20.520 17° 20.980	21.000 14° 30′ 21.000					
E_1 α E_2	41	18.120 17° 21.380	18.600 17° 21.400	19.080 17° 21.420	19.560 17° 21.440	20.040 17° 21.460	29.520 17° 21.480	21.000 14° 30′ 21.500	21.500 14° 30′ 21.500				
E_1 α E_2	42	18.120 17° 21.880	18.600 17° 21.900	19.080 17° 21.920	19.560 17° 21.940	20.040 17° 21.960	20.520 17° 21.980	21.000 14° 30′ 22.000	21.500 14° 30′ 22.000	22.000 14° 30′ 22.000			
E_1 α E_2	43	18.120 17° 22.380	18.600 17° 22.400	19.080 17° 22.420	19.560 17° 22.440	20.040 17° 22.460	20.520 17° 22.480	21.000 14° 30′ 22.500	21.500 14° 30′ 22.500	22.000 14° 30′ 22.500	22.500 14° 30′ 22.500		
E_1 α E_2	44	18.120 17° 22.880	18.600 17° 22.900	19.080 17° 22.920	19.560 17° 22.940	20.040 17° 22.960	20.520 17° 22.980	21.000 14° 30′ 23.000	21.500 14° 30′ 23.000	22.000 14° 30′ 23.000	22.500 14° 30′ 23.000	23.000 14° 30′ 23.000	
E_1 α E_2	45	18.120 17° 23.380	18.600 17° 23.400	19.080 17° 23.420	19.560 17° 23.440	20.040 17° 23.460	20.520 17° 23.480	21.000 14° 30′ 23.500	21.500 14° 30′ 23.500	22.000 14° 30′ 23.500	22.500 14° 30′ 23.500	23.000 14° 30′ 23.500	23.500 14° 30′ 23.500

Legend: E_1 = outside radius of pinion, inches. α = pressure angle of basic rack. E_2 = outside radius of gear, inches. The working depths vary with the pressure angles as follows:

Pressure angle:	25 deg.	22 deg. 30 min.	20 deg.	17 deg.	14 deg. 30 min.
Working depth:	1.800 in.	2.000 in.	2.200 in.	2.200 in.	2.200 in.

MAAG GEARS

The idea of a system of variable-center distance gears to be generated by a single basic rack is not new. In Germany, Hoppe proposed a system of interchangeable gears where the pressure angle was variable, the normal pitch constant, and the center distances could be taken from a table. Hoppe's system was described by Kammerer in *Zeitschrift des Vereines Deutscher Ingenieure*, 1903, page 885. Broadly speaking, Hoppe's suggestions are similar in many ways to the principles which were later followed by Maag.

Maag was probably the first manufacturer to produce commercially involute gears designed to obtain the full benefits of the involute form. The design of these gears follows, in general, the same principles that have been followed in the foregoing systems. A series of range cutters is employed, with which gears to run at standard- or special-center distances can be produced with very favorable and well-balanced tooth designs. These gears have been extensively used in Europe for the past 20 years and in the United States for the past 10 years.

The table of outside radii and pressure angles for the range-cutter, proportional-center distance system is for use on pairs of gears only. When a train of more than two gears is involved, one of two courses should be followed. If the center distances can be varied, the gears can all be cut with the $14\frac{1}{2}$-deg. cutter, and the center distances be established according to the table of variable-center distance gears. For example, if the train of gears had tooth numbers of 35, 22, 40, and 90, respectively, we should obtain the following center distances and tooth depths from Table XII of variable-center distance gears.

Combination	Center distance, inches	Tooth depth, inches
$^{35}/_{22}$	28.9715	2.1498
$^{22}/_{40}$	31.3795	2.1703
$^{40}/_{90}$	65.0000	2.2000

In such cases, each gear should be made with the minimum tooth depth specified for any of its meshes, so as not to reduce the clearances. This would give us the following:

Number of teeth	35	22	40	90
Tooth depth, inches	2.1498	2.1498	2.1703	2.2000

When the center distances are fixed and are proportional to the numbers of teeth, it is best to make the addenda of all the gears in the train equal and to use the cutter that is required for a pair of the smallest gears in the train. For example, if the train of gears had tooth numbers of 35, 22, 40, and 90, respectively, the 20-deg. cutter should be used, as that is the one required for a pair of 22-tooth gears. In this case, the addenda of all the gears

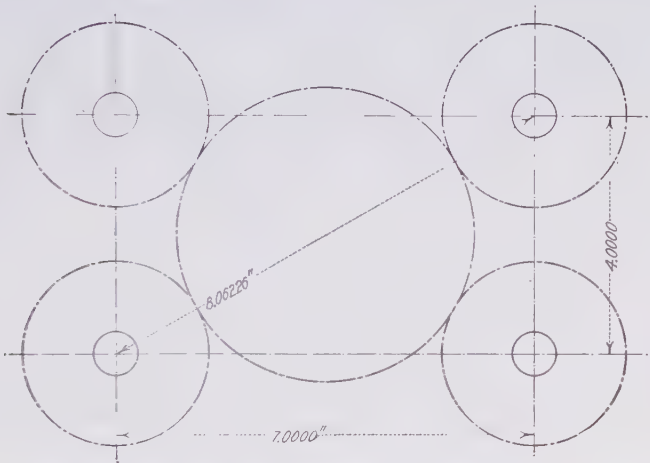


FIG. 80.—Gear train for multiple-spindle drilling attachment.

in the train would be equal to 1.0000 in. If the smallest gear in the train had 16 teeth, the $22\frac{1}{2}$-deg. cutter should be used, and the addenda on all the gears would be 0.9000 in.

Conditions arise where a pair or train of gears is required to operate on special-center distances. For example, we will assume that we wish to make a multiple-spindle drilling attachment to drill holes, as shown in Fig. 80. There will be four spindles, all to be driven by a single gear in the center. The diagonal distance from center to center of these holes is 8.06226 in. One-half of this, or 4.03113 in., will be the center distance between the driving gear and the driven ones on the drill spindles. We will assume that 10-d.p. gears will be strong enough for this drive. The first step would be to reduce this center distance to the corresponding one of 1-d.p. gears, because it is simpler

always to calculate on the basis of 1-d.p. gears. This is done in this example by multiplying by 10, which gives us 40.3113 in. as the center distance for 1-d.p. gears. The nearest smaller sum of teeth for this center distance is 80. In this example, we will therefore make the central gear with 50 teeth, and the gears on the spindles with 30 teeth.

Referring to the table of proportional-center distance gears, Table XIV, we find that these gears would ordinarily be cut with the 17-deg. cutter. We will therefore use this same cutter for these gears.

The next step is to determine the proper root radii for these gears that are to mesh at a special-center distance. Referring to Prob. 12, in Chap. III, we have

$$C_1 = \frac{C_2 \cos \alpha_2}{\cos \alpha_1} \qquad \text{(see Eq. (51))}$$

$$G + g = C_1 - 2F + \frac{\cot \alpha_1}{2} \left[2C_1 \left(\frac{\pi}{N + n} + \operatorname{inv} \alpha_2 - \operatorname{inv} \alpha_1 \right) - c.p. \right] \qquad \text{(see Eq. (52))}$$

Where α_1 = pressure angle of cutter of basic-rack form
α_2 = pressure angle of gears when meshed
C_1 = center distance with pressure angle of α_1
C_2 = center distance with pressure angle of α_2
F = addendum of basic-rack form cutter, including clearance
N = number of teeth in gear
n = number of teeth in pinion
$c.p.$ = circular pitch of rack
G = root radius of gear
g = root radius of pinion

In this example, we have

$$C_2 = 40.3113 \text{ in.}$$
$$\alpha_1 = 17 \text{ deg.}$$
$$N = 50 \text{ teeth}$$
$$n = 30 \text{ teeth}$$
$$F = 1.2000 \text{ in.}$$

If these gears were to run at a pressure angle of 17 deg., their center distance would be 40.000 in., whence,

$$C_1 = 40.000 \text{ in.}$$

Transposing Eq. (51), to solve for α_2, we have

$$\cos \alpha_2 = \frac{C_1 \cos \alpha_1}{C_2} = \frac{40 \times 0.9563}{40.3113} = 0.9489$$

whence,

$$\alpha_2 = 18 \text{ deg. } 24 \text{ min. and inv } \alpha_2 = 0.011515$$
$$\text{inv } \alpha_1 = 0.009025$$

Solving Eq. (52), we have

$$G + g = 40 - 2.4000 +$$
$$1.6354\left[80\left(\frac{3.1416}{80} + 0.011515 - 0.009025\right) - 3.1416\right]$$
$$G + g = 37.9258 \text{ in.}$$

With a center distance of 40.3113 in., the distance between the root radii will be $40.3113 - 37.9258 = 2.3855$ in. This distance includes the working depth plus two clearances. With a clearance of one-tenth the working depth, we have

$$1.200 \times \text{working depth} = 2.3855 \text{ in.}$$
$$\text{working depth} = 1.9879 \text{ in.}$$
$$\text{clearance} = 0.1988 \text{ in.}$$
$$\text{tooth depth} = 2.1867 \text{ in.}$$

Referring again to Table XIV, of proportional-center distance gears, we find that the outside radius of a 30-tooth gear that is to mesh with a gear of 40 teeth or more is 16.2000 in. We will use this same outside radius for this special gear, as the smaller of a pair always has the most sensitive tooth profile, so that it is safest to make any necessary variation on the larger gear. The corresponding root radius is 14.0133 in. for the 30-tooth gear. Subtracting this from the sum of the root radii, we obtain 23.9125 in. for the root radius of the 50-tooth gear. Thus, we have the following values for this pair of gears:

	1 d.p.		10 d.p.	
	Pinion	Gear	Pinion	Gear
Number of teeth	30	50	30	50
Outside radius, inches	16.2000	26.0992	1.6200	2.6099
Root radius, inches	14.0133	23.9125	1.4013	2.3912
Tooth depth, inches	2.1867		0.2187	
Center distance, inches	40.3113		4.0311	
Pressure angle of cutter	17 deg.			

With the equations given in Chaps. II and III, and the many examples given there and in these last few chapters sufficient

information is available to enable any pair or train of gears to be calculated, so that further examples should be unnecessary.

Pinions of Small Tooth Numbers.—The smallest pinion considered thus far is one with 10 teeth. Smaller pinions can be made, but they have been purposely omitted from all previous discussion, because they should not be generally used for the transmission of power. Exceptional conditions are met occasionally, where the use of smaller pinions seems advisable. We shall therefore give these smaller pinions consideration at this time.

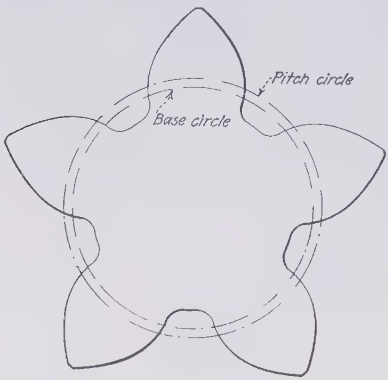


Fig. 81.—Modified 5-tooth pinion.

Theoretically, the smallest involute spur pinion that can be made with symmetrical teeth and give an overlap, or contact of one tooth interval or more, is one with five teeth. We will therefore direct our attention to a 5-tooth pinion.

When the numbers of teeth in the gears are small, the duration of contact is the limiting factor. In order to secure a line of action of sufficient length, relatively high pressure angles must be used, because the greater the pressure angle, the smaller the base circle, the longer the length of the common tangent, and the shorter the length of the normal pitch. Extra length of contact is secured at the expense of sensitive tooth profiles and excessive sliding conditions.

The fewer the number of teeth in a gear, the greater is the care required to select the most favorable pressure angle and tooth proportions to secure sufficient contact without losing almost every valuable property of the involute curve.

In order to obtain any appreciable contact on a 5-tooth pinion, it is necessary to use its involute profile practically to the base circle. This means a very sensitive and troublesome tooth profile. Furthermore, the depth of the tooth on such a pinion is limited, because it soon comes to a point at the tip. Although it is possible theoretically to make a pair of 5-tooth pinions that will mesh together with an overlap on the contact, to accomplish this would require the use of special cutters with a very high pressure angle. We will not attempt this but will

TABLE XV.—OUTSIDE AND ROOT RADII OF SMALL PINIONS MESHING WITH GEARS OF THE 22½-DEG., BASIC-RACK, PROPORTIONAL-CENTER DISTANCE SYSTEM

Number of teeth in gear		Number of teeth in pinion				
		5	6	7	8	9
Outside radius............ Root radius...............	Pinion	3.8600 1.9350	4.3650 2.3650	4.7950 2.7950	5.2250 3.2250	5.6550 3.6550
Outside radius............ Root radius...............	15	....	....	....	....	8.1450 6.1450
Outside radius............ Root radius...............	16	....	....	....	8.5750 6.5750	8.6450 6.6450
Outside radius............ Root radius...............	17	....	....	9.0050 7.0050	9.0750 7.0750	9.1450 7.1450
Outside radius............ Root radius...............	18	....	9.4350 7.4350	9.5050 7.5050	9.5750 7.5750	9.6450 7.6450
Outside radius............ Root radius...............	19	9.8650 7.8650	9.9350 7.9350	10.0050 8.0050	10.0750 8.0750	10.1450 8.1450
Outside radius............ Root radius...............	20	10.3650 8.3650	10.4350 8.4350	10.5050 8.5050	10.5750 8.5750	10.6450 8.6450
Outside radius............ Root radius...............	21	10.8650 8.8650	10.9350 8.9350	11.0050 9.0050	11.0750 9.0750	11.1450 9.1450
Outside radius............ Root radius...............	22	11.3650 9.3650	11.4250 9.4350	11.5050 9.5050	11.5750 9.5750	11.6450 9.6450
Outside radius............ Root radius...............	23	11.8650 9.8650	11.9350 9.9350	12.0050 10.0050	12.0750 10.0750	12.1450 10.1450
Outside radius............ Root radius...............	24	12.3650 10.3650	12.4350 10.4350	12.5050 10.5050	12.5750 10.5750	12.6450 10.6450
Outside radius............ Root radius...............	25	12.8650 10.8650	12.9350 10.9350	13.0050 11.0050	13.0750 11.0750	13.1450 11.1450
Outside radius............ Root radius...............	26	13.3650 11.3650	13.4350 11.4350	13.5050 11.5050	13.5750 11.5750	13.6450 11.6450
Outside radius............ Root radius...............	27	13.8650 11.8650	13.9350 11.9350	14.0050 12.0050	14.0750 12.0750	14.1450 12.1450
Outside radius............ Root radius...............	28	14.3650 12.3650	14.4350 12.4350	14.5050 12.5050	14.5750 12.5750	14.6450 12.6450
Outside radius............ Root radius...............	29	14.8650 12.8560	14.9350 12.9350	15.0050 13.0050	15.0750 13.0750	15.1450 13.1450
Outside radius............ Root radius...............	30	15.3650 13.3650	15.4350 13.4350	15.5050 13.5050	15.5750 13.5750	15.6450 13.6450
Outside radius............ Root radius...............	31	15.8650 13.8650	15.9350 13.9350	16.0050 14.0050	16.0750 14.0750	16.1450 14.1450
Outside radius............ Root radius...............	32	16.3650 14.3650	16.4350 14.4350	16.5050 14.5050	16.5750 14.5750	16.6450 14.6450
Outside radius............ Root radius...............	33	16.8650 14.8650	16.9350 14.9350	17.0050 15.0050	17.0750 15.0750	17.1450 15.1450
Outside radius............ Root radius...............	34	17.3650 15.3650	17.4350 15.4350	17.5050 15.5050	17.5750 15.5750	17.6450 15.6450
Outside radius............ Root radius...............	35	17.8650 15.8650	17.9350 15.9350	18.0050 16.0050	18.0750 16.0750	18.1450 16.1450

Tooth depth = 2.0000 in. Working depth = 1.8000 in.

consider a five-tooth pinion that is generated with a 22½-deg. cutter. This cutter has a working depth of 1.8000 in. and a clearance of 0.2000 in. We will make the root radius close to the undercut limit, and if the tip of the tooth becomes pointed, we will reduce its outside radius until this tip is at least 0.1000 in. on a pinion of 1 d.p. Thus, we have the following:

R = 2.5000 in. = pitch radius at 22½-deg. pitch line
α = 22 deg. 30 min. = pressure angle of cutter
F = 1.2000 in. = addendum of cutter including clearance
f = 0.2000 in. = clearance
a = radius of base circle
A = undercut limit

$$a = R\cos\alpha \qquad = 2.3097 \text{ in.}$$
$$A = R\cos^2\alpha - f = 1.9339 \text{ in.}$$

We will round off this undercut limit to 1.9350 in. for the root radius, whence,

H = 1.9350 in. = root radius
T_1 = 2.0140 in. = thickness of tooth on 22½-deg. pitch line.

With a tooth depth of 2.0000 in., the outside radius of this pinion would be 3.9350 in. We will first check the thickness of the tooth at this radius. From Chap. III, we have

$$\cos\alpha_2 = \frac{r_1\cos\alpha_1}{r_2} \qquad \text{(see Eq (39))}$$
$$T_2 = 2r_2\left(\frac{T_1}{2r_1} + \operatorname{inv}\alpha_1 - \operatorname{inv}\alpha_2\right) \qquad \text{(see Eq. (40))}$$

In this example,
$$\alpha_1 = 22 \text{ deg. } 30 \text{ min.}$$
$$r_1 = 2.5000 \text{ in.}$$
$$r_2 = 3.9350 \text{ in.}$$
$$T_1 = 2.0140 \text{ in.}$$

whence,

$\cos\alpha_2 = 0.58696$ $\alpha_2 = 54$ deg. 4 min. $\operatorname{inv}\alpha_2 = 0.43606$ $\operatorname{inv}\alpha_1 = 0.02151$ and $T_2 = -0.0925$

Hence, this tip becomes pointed before it reaches the specified outside radius. We will therefore reduce this outside radius to 3.8600 in. and obtain the following conditions:

$r_2 = 3.8600$ in. $\cos\alpha_2 = 0.59836$ $\alpha_2 = 53$ deg. 15 min. $\operatorname{inv}\alpha_2 = 0.40982$ and $T_2 = 0.1119$ in.

TABLE XVI.—CONTACT IN TERMS OF TOOTH INTERVALS BETWEEN SMALL PINIONS AND GEARS OF THE 22½-DEG., BASIC-RACK, PROPORTIONAL-CENTER DISTANCE SYSTEM

Number of teeth in gear	Number of teeth in pinion				
	5	6	7	8	9
15	.....	.....	.....	.....	1.213
16	.....	.....	.....	1.188	1.216
17	.....	.....	1.156	1.191	1.219
18	.....	1.115	1.158	1.194	1.222
19	1.035	1.117	1.160	1.196	1.225
20	1.036	1.119	1.161	1.198	1.228
21	1.037	1.120	1.163	1.200	1.231
22	1.038	1.122	1.165	1.202	1.233
23	1.039	1.123	1.167	1.204	1.236
24	1.039	1.124	1.169	1.206	1.239
25	1.040	1.125	1.170	1.208	1.241
26	1.041	1.126	1.172	1.210	1.243
27	1.042	1.127	1.173	1.212	1.245
28	1.042	1.128	1.174	1.214	1.247
29	1.043	1.129	1.175	1.215	1.249
30	1.044	1.130	1.176	1.216	1.251
31	1.044	1.130	1.177	1.218	1.253
32	1.045	1.131	1.178	1.219	1.255
33	1.045	1.132	1.179	1.220	1.256
34	1.046	1.132	1.180	1.221	1.257
35	1.046	1.133	1.180	1.222	1.258
36	1.047	1.134	1.181	1.223	1.259
37	1.047	1.134	1.182	1.224	1.260
38	1.048	1.135	1.182	1.225	1.261
39	1.048	1.135	1.183	1.225	1.262
40	1.049	1.136	1.184	1.226	1.263
Rack.........	1.064	1.157	1.212	1.264	1.312

We now have the following values for the five-tooth pinion:

$$
\begin{aligned}
E_1 &= 3.8600 \text{ in.} = \text{outside radius}\\
R_1 &= 2.5000 \text{ in.} = 22\tfrac{1}{2}\text{-deg. pitch radius}\\
a_1 &= 2.3097 \text{ in.} = \text{radius of base circle}\\
H_1 &= 1.9350 \text{ in.} = \text{root radius}
\end{aligned}
$$

The correction, or increase in the root radius of this pinion, is equal to 0.5350 in. The smallest gear that can have a corresponding decrease in the root radius without undercutting is

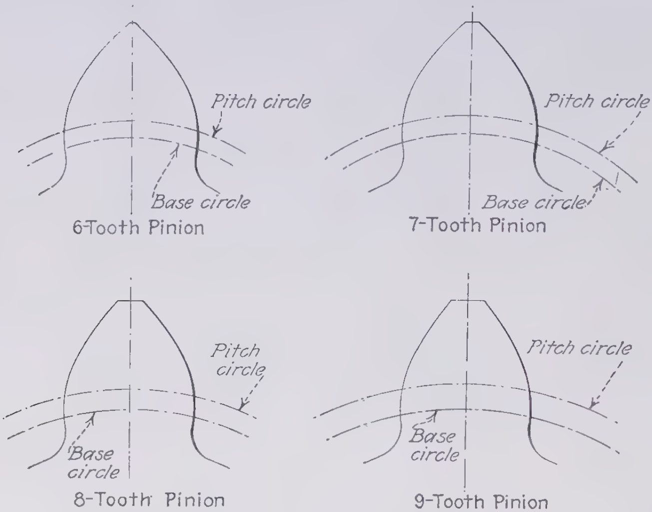


FIG. 82.—Tooth forms of six-, seven-, eight- and nine-tooth pinions.

one with 19 teeth. This gear, to run with the five-tooth pinion on a proportional-center distance, would have the following values:

$$
\begin{aligned}
E_2 &= 9.8650 \text{ in.}\\
R_2 &= 9.5000 \text{ in.}\\
H_2 &= 7.8650 \text{ in.}\\
a_2 &= 8.7769 \text{ in.}\\
C &= 12.0000 \text{ in.}
\end{aligned}
$$

The contact between this pair of gears would be as follows:

$$
\begin{aligned}
\sqrt{E_1^2 - a_1^2} &= 3.0927\\
\sqrt{E_2^2 - a_2^2} &= 4.5038\\
C \sin \alpha &= 4.5922\\
Pn &= 2.9025 \text{ in.}\\
\text{Contact} &= 1.035 \text{ tooth intervals}
\end{aligned}
$$

The sliding conditions would be as follows:

Number of teeth	5	19
Maximum radius of curvature of active profile	3.0927 in.	4.5038 in.
Minimum radius of curvature of active profile	0.0884 in.	1.4995 in.
Specific sliding, addendum	+ 0.87	+0.92
Specific sliding, dedendum	−12.40	−6.83

It will be seen from the foregoing that the tooth profiles on such small pinions are, of necessity, very sensitive. In Fig. 81, this pinion is shown.

In a similar manner, values for other small pinions of six, seven, eight, and nine teeth have been established. These are tabulated in the accompanying Table XV. Contact conditions in terms of tooth intervals for these small pinions are also given in Table XVI. Figure 82 shows the forms of the teeth of the six-, seven-, eight-, and nine-tooth pinions.

These pinions are suitable for use as motor-starting pinions and for drives where minimum size is the controlling factor. They should be used only to meet extreme conditions.

SECTION II

GEAR TEETH IN ACTION

CHAPTER VI

GEAR TEETH IN ACTION

Satisfactory gears must transmit power smoothly, with a minimum of vibration and noise, and must also have a reasonable length of useful life. In order to accomplish these ends, several essential requirements must be met. These requirements are of varying degrees of importance, depending largely upon the service which the gears are to give. Some of them are requirements of the gears themselves, others have to do with their mounting and care in operation.

One word of caution at the outset: No improvement in the design of the tooth forms alone will outweigh the importance of careful and painstaking workmanship in both the production and the mounting of the gears. As a matter of fact, the better tooth forms deserve to receive more care and attention than the poorer forms. A poor design of any mechanism that proves successful is usually a triumph of good workmanship over poor design.

NOISE OF GEARS

It would be well to state at once that no metal gears in operation are absolutely noiseless. Quietness is a relative term. The most accurate gears when running under load at any appreciable speed will develop a certain amount of sound.

The noise of gears is of many kinds. It ranges from the unobtrusive hum of the better gears to rumbles and squeals of varying pitches and intensities of the poorer grade. The exact cause of all conditions of noise are not, as yet, fully known. One fact, however, is certain: Excessive noise is evidence of improper conditions somewhere in the mechanism. In many respects, it is not a matter of why gears are noisy but rather why they are ever quiet. Certain types of defects in the gears, however, develop certain characteristic noises.

Noise is relative rather than absolute. It may be defined as an unpleasant or objectionable sound. This is certainly a proper

definition of the noise of gears. There is only one sure method of reducing the amount of sound produced by gears. That is to increase the accuracy and smoothness of profile in their production. More favorable gear-tooth forms may help some, but this improved design must be coupled with high-class workmanship in order to secure the best results.

The search has been made for many years, and still continues, for some form of modification of gear-tooth profiles that will obviate the need of extreme accuracy. This search has been fruitless in the past and probably always will be. If such a modified form were discovered, it probably would also be found that the forming of this modification must be done with great accuracy and that the spacing of the teeth must be as nearly perfect as on other forms to secure quiet running, while the difficulties of control would be even greater than at present. This does not mean that all modifications of gear-tooth profiles are always undesirable. Certain slight modifications can be made to advantage at times that tend to minimize the effects of other small errors. The attempt should always be made to keep such modifications to a minimum. In most cases, this modification should be in the nature of a tolerance, that is, the direction of permissible errors on the tooth profiles should be in the direction that avoids edge contact at the beginning of mesh. The idea that if a little modification is good, more is better, does not prove out in practice.

Four general characteristic sounds are produced by gear teeth in operation under load. One is an intermittent clicking or steady growl caused by poor spacing or irregularly formed tooth profiles. The second is a pulsating growl or run-out sound caused by eccentricity of the gears. The third is a high-pitched squeal caused by rough tooth surfaces. The only remedy for all these three is better workmanship. The fourth sound is a tone that depends upon the pitch and speed of the gears. This sound approaches a musical tone with the higher pitch-line velocities, the pitch of this tone depending upon the number of tooth engagements per second. If this tone is steady, it indicates uniformity in the gear-tooth profiles. If it varies in intensity, it is evidence of a variation in the meshing conditions. Such variations may be caused by faulty tooth profiles or spacing, variation in the load transmitted, or springing of the gear shafts under the load, and the like.

Uniformity of Tooth Profiles and Spacing.—It is evident that in order to secure smooth transmission of power by means of gear teeth, both the profiles and the spacing of the teeth must be accurate. Gear teeth are nothing more than cams acting against each other. An error in either the form or position of these cams will cause a change of greater or less amount in the velocity of the driven gear. This change will occur in a very short space of time as the inaccurate teeth come into contact. The greater the speed of the gears, the shorter the time in which this change will take place.

Such variations in the uniformity of the motion transmitted will develop noises of varying kinds and intensities, depending upon the nature and extent of the error. Practically all gears are made with a certain amount of backlash between each tooth and the corresponding space on the mating gear. Too great an error in the profile will cause an intermittent, metallic rattle as the momentum of the driven gear, when speeded up by a faulty profile, throws the non-active profile of its tooth against the non-active profile of the tooth in the driving gear. This condition is particularly noticeable when the gears are running idle or under but slight loads. Such errors in the gear-tooth profiles are also serious as regards the strength and wear of the gear teeth. Correct profiles, both as to form and spacing, are of even more importance for strength and long wear than they are for quiet running. There are many cases where the noise is of secondary importance, but there are few places where strength and durability are not of primary importance.

On involute gears, an error in either the tooth profile or spacing will make a difference in the normal pitch. This normal pitch should be uniform on any gear, and it should be identical on mating gears if the most satisfactory results are to be obtained. If the normal pitch on each gear is uniform but the normal pitch on one gear is not identical with that of its mating gear, a steady growl will be heard, the pitch of the sound being dependent upon the speed of the gears, and its intensity upon the extent of the difference. If this normal pitch is irregular, a grumble of varying intensity will develop. This last is one of the most disagreeable of gear noises.

Even with the greatest care, some inaccuracies will always be present. The second best thing to absence of all errors is to have such errors as are unavoidable of such a nature that they

cause the least annoyance. Edge contact at the beginning of mesh is always dangerous. It not only causes a sudden change in velocity with the attendant noise but also tends to gouge out the profile of the mating tooth. Whatever departure, therefore, from theoretically correct profiles is present should be in a direction that will avoid edge contact at the beginning of mesh. To accomplish this end requires a modification of the theoretical profile. The amount of such modification is dependent upon the extent of the errors that may be present. This modification may be secured in either one of two ways: First, the tip of the profile of the tooth of the driving gear may be cut back slightly, or "eased off;" or, second, the normal pitch of the driving gear may be made slightly greater than that of the driven gear. When a normal-pitch difference is used for this purpose, the amount of difference should be used as a tolerance. Both gears should have the same basic normal pitch, but the tolerance on the length of the normal pitch of the driving gear would be plus, while the tolerance for the driven gear would be minus.

The base circle of the involute profiles should be concentric with the center on which the gear actually operates. This is a matter that requires great care while the gears are being cut. It is good practice to true the outside diameters of gear blanks by turning or grinding on an arbor before cutting the teeth, so that the concentricity of the set-up for cutting the teeth can be accurately checked. This outside diameter of itself is but of little importance. It is, however, the most convenient and therefore the most important reference surface for securing concentricity of gear-tooth profiles.

An eccentric gear develops a sound of rising and falling intensity, which is readily distinguished from the other noises. This noise is usually a disagreeable one. Not only is the noise objectionable, but also the irregular motion transmitted by eccentric gears is seldom desirable.

Effect of Eccentricity on Gear Noises.—An analysis of the action of purely eccentric involute gears with large or small eccentricities shows that they will give a varying but continuous action. Each succeeding tooth takes over the load from the preceding one smoothly, without a blow. With uniform motion of the driving gear, the velocity of the driven gear will vary according to a pure sine curve. It is not an intermittent action such as develops from spacing errors. Thus, the eccentricity of

accurate involute gears does not introduce the effect of spacing errors and has but relatively little effect on the strength of the gears, because the accelerations caused by such eccentricity take place in a relatively long length of time. Eccentricity of gears is, therefore, objectionable primarily because of the resulting varying sound in operation and the irregular motion transmitted.

This smoothness of action of eccentric gears requires a sufficiently long arc of contact to secure an overlap at all positions of mesh and also a large enough tooth load to keep the teeth of the two gears in contact. Otherwise, a rattling noise will also be present in addition to the characteristic run-out sound.

Eccentricity is extremely objectionable on change gears used for accurate dividing or screw cutting. Even very small eccentricities have a marked effect on the accuracy of the final result. Particular pains should always be taken in the cutting of change gears to keep the amount of eccentricity to a minimum.

For quiet operation, it is most essential that the surfaces of the tooth profiles of the gears be as smooth as it is possible to make them. A roughness on these profiles that is hardly noticeable to the eye will develop considerable noise when the gears are run under load at any appreciable speed. The sound caused by rough tooth profiles is a distinctive high-pitched scream, which varies in its intensity more than in the pitch of the tone as the speed and load is increased. This sound is very penetrating and, hence, is all the more objectionable.

Rough surfaces on gear-tooth profiles not only are detrimental to quiet operation but also are responsible for rapid wear. When this roughness is very slight, it sometimes smooths down in a short time, when the gears are operated under load, without any appreciable wear's taking place, so that much of this noise is eliminated. The amount of the roughness that can be eliminated in this manner, however, is slight. If it is excessive, the accuracy of the tooth profiles will have been destroyed by wear, resulting in greater profile and spacing errors, with all their objectionable features, before the tooth profiles become smoothed down. In other cases, this original roughness is never eliminated but rather gets worse with time and wear. This condition seems to be particularly true when two soft-steel gears run together. The nature of this material is such that it seems to abrade and rough up rather than polish down.

"The Music of the Gears"

Several years ago the writer was discussing various gear problems with Charles H. Logue, and the discussion turned to the subject of the musical tones in the sounds produced by gears. Most of the following material on this subject is taken from notes furnished the writer by Mr. Logue.

The extent to which the science of musical sounds may enter into machine design is but slightly appreciated. A successful engineer or mechanic has a certain instinct in this direction—he must have—but it does not appear that any definite connection is generally recognized. While the study both of dynamics and of sound are included under the same general heading—Physics—it seems, in practice, a far cry from the study of machine design, for instance, to the study of music. There is, nevertheless, a close connection.

"The distinction between music and noise is, generally speaking, a distinction between the agreeable and the disagreeable."

For commercial gear drives, where all noise cannot be eliminated, there is a demand for some degree of quietness; that is, what sound is present should be unnoticeable or agreeable. These less noticeable sounds are generally found in the lower pitches or tones. In usual gear design, the attempt is made, at times, to reach pitches so high that they are indistinguishable. The usual results are poor, although there are installations where it is good practice.

Noise in the operation of gears is caused primarily by errors in individual tooth action. The ear is capable of hearing sounds arising from 32 to 38,000 vibrations per second. Thus, when the number of tooth engagements per second exceeds 32, the result is a continuous noise, which may be recognized as a tone whose note or vibration period is designated by the number of tooth engagements made per second. In case these vibrations should exceed 38,000 per second, no tone would be heard from this cause, but such speeds are beyond the ordinary range of gear practice. This may tend to explain, however, why gear units that are capable of running quietly at a speed of 5,000 ft. per minute may be raised to any higher practical speed, say 8,000 to 12,000 ft. per minute, without additional noise.

The starting point of the musical scale, that is, the number of vibrations per second for the key note, is entirely a matter of

convention. In practice, there is a great lack of agreement. The ratios between the various tones, however, are constant, regardless of the exact number of vibrations per second selected as the starting point. The arbitrary scale of pitch in common use, and typified by the white keys of a piano, is represented by the following ratios, which are also very close to the actual number of vibrations per second ordinarily used:

Note	Frequency	Note	Frequency
c	128.0	c′	256.0
d	144.0	d′	288.0
e	160.0	e′	320.0
f	170.6	f′	341.3
g	192.0	g′	384.0
a	213.3	a′	426.6
b	240.0	b″	480.0
		c″	512.0

Thus, if a pair of gears revolved at such a speed that 256 tooth contacts were made each second, the pitch of the sound would be approximately that of middle C on the keyboard of a piano.

Resonance.—When we are dealing with a single pair of gears, this study is primarily one of resonance, the gears being the generator or source of the vibrations, and the carrier or case containing the gears, the resonator. We have also to consider the natural tone frequency of the gears themselves. This will be considered later when we discuss the gear blanks. If the case containing the gears has a natural tone frequency of, say, 256 vibrations per second, it will respond to the tone of the gears when their speed is such that the number of contacts made by the gear teeth per second is 256, or in that neighborhood. If the tone frequency of either of the gear blanks has this same natural frequency, this will also tend to enlarge upon the sound produced. Under these conditions, the case containing the gears becomes an amplifier, so that the result is a far greater volume of sound than originally started from the gear teeth themselves. It should be understood that the resonator can produce no noise which does not originate from some other source; in this example, the gears themselves. The study of violin construction resolves itself into the production of a resonator (violin body) that will respond equally to all tones. The ideal gear case would be one that would not respond to any tone.

When two or more pairs of gears are engaged, we have to consider consonance in addition to the study of resonance. The combination of tones produced by the various gears may be discordant or harmonious, according to the ratio of their tone frequencies to each other. In general, the difference between consonance and dissonance, or between harmony and discord, or between music and noise, is that noise is unpleasant or irritating while music is heard either with pleasure or with indifference. Musical tone combinations are, of course, to be desired, as they are not only more pleasing, or softer, but they are also more rapidly blanketed or dissipated.

Harmonious Ratios.—The general law governing consonance and dissonance, as stated by Dr. Helmholtz, in his "Sensations of Tone," is simplicity itself:

> Generally those tones, and only those tones, harmonize whose fundamental tones bear to one another ratios expressed by small numbers; and the smaller the numbers which express the ratios of vibrations, the more perfect is the harmony of the two sounds.

For example, the following ratios give pleasing combinations of tones:

Ratio	Tone interval
1:1	Single tone
1:2	Octave
1:3	Octave and perfect fifth
1:4	Two octaves
1:5	Two octaves and major third
1:6	Two octaves and perfect fifth
2:3	Perfect fifth
2:5	Octave and major third
3:4	Perfect fourth
3:5	Major sixth
4:5	Major third
5:6	Minor third

Except when the ratio gives an even octave, such as 1:8, three octaves; 1:16, four octaves; etc.; when the figure that expresses the exact ratio becomes larger than six, we begin to run into discords. For example, 8:9 is a major second and is more discordant than harmonious.

For a definite example, we will consider the gears in an automobile transmission case. All of the gears on the countershaft make the same number of revolutions. The combination of tones that result from the tooth engagements of any two pairs of

gears in such a case will have a frequency ratio the same as the ratio of the numbers of teeth in the gears on the countershaft. By keeping the numbers of teeth in these gears to harmonious ratios, we can avoid certain unpleasant sounds. For example, if the constant-mesh gear has 30 teeth and the second-speed countershaft gear has 25 teeth, the ratio will be 5:6 and their combined sound will produce a minor third. If the second-speed gear has 24 teeth, the ratio will be 4:5, and their combined tone will produce a major third. Both of these combinations are harmonious. The low-speed gear could have 15 teeth, which would give a ratio of 1:2, or an octave tone interval, while the reverse gear could have 12 teeth, which would give a ratio of 2:5 and a tone interval of an octave and a major third.

Some European automobile transmissions are now made with all of the gears on the countershaft with the same number of teeth, changing the pitch of the gears to secure the desired reduction ratios, in order to have but a single tone due to the tooth engagements. It is also of interest to note in this connection that the easiest transmission in the writer's experience to keep quiet was one in which the tooth numbers on three of the four countershaft gears gave ratios corresponding to harmonious combinations; and, also, that the most difficult one to deal with had tooth numbers that gave ratios of very discordant combinations of sound. The tooth numbers alone were not entirely responsible, in either case, but they were unquestionably important contributing factors.

Loudness depends upon the amplitude of vibrations, density of the medium in which the vibrations are created, and distance. With gears, the amplitude will depend primarily upon the error in pitch velocity, that is, approximately upon the percentage of error; the medium and the distance are factors over which we have little or no control.

An equal volume of sound may be produced by two pairs of gears, one having a low pitch with high amplitude and the second pair having a high pitch with low amplitude. It then becomes a question of which is the least objectionable or the ability of the installation to cover up or blanket the sound. Of the two, the first is the easier to muffle; high notes are more penetrating.

In case the gears are made for machine-tool drives, the high-pitched gears are often preferable, despite the character of the sound, because each vibration is usually plainly distinguishable

as "gear-tooth marks" on the piece being machined. Each installation has its own peculiar problems in this respect.

A distinction must be made between the smoothness of operation and the noise of operation. A fine circular pitch may increase the smoothness of operation, but it may also increase the noise of operation. The answer lies in the relative accuracy and tooth design.

INFLUENCE OF GEAR BLANK AND CARRIER DESIGN

The shape of the gear blank itself often has a marked effect on the relative quietness of the gears. A blank that has the general properties of a bell will pick up and sustain a sound of its own vibration frequency that is created either by the engagement of the gear teeth or by some other outside agency and will oftentimes amplify such sounds considerably. In many cases, the introduction of ribs on such blanks will greatly reduce their resonance. Again, a gear made in two parts, such as a forged-steel rim mounted on a cast-iron or a cast-steel core or one split in two sections, will oftentimes operate more quietly than a solid gear, as vibrations are dampened or absorbed by a machined joint, either in the gears themselves or in their case or carrier. A gear made of built-up sections is often ideal from this point of view. Quietness of gear operation depends, to a large extent, upon the ability of the construction of both the gears and their carrier to dampen or absorb vibrations set up by the action of the gear teeth.

On the other hand, gears that are to operate at relatively high pitch-line velocities must usually be solid. Steel rims shrunk on cast cores are used to some extent, but continued operation tends to loosen the steel rims. The same tooth engagements that cause the sound also tend to peen out the gear ring and thus make it larger. When a rim is shrunk on a core for use as a gear that is to operate at high pitch-line velocities, it should be secured from turning by other means than a tight fit due to shrinkage.

In order to obtain the best results, gears must be rigidly mounted so as to hold the teeth in their proper relationship to each other when they are in operation. The shafts or studs that carry spur gears must be parallel. The exact-center distance for involute gears is of secondary importance; the alignment is far more important. Gears should be carried as closely to a rigid support as possible and should not be mounted in the middle of a

long flexible shaft insufficiently supported nor on the end of a shaft that projects considerably from its bearing. Noisy gear conditions are not always due to improper gears. When the design of the mounting is at fault, even perfect gears perform unsatisfactorily.

Some sound is inevitable when power is transmitted through gear teeth. The volume of this sound may be small, yet the size and shape of the gear case may be such that much more sound will be heard outside the case than originally started from the inside. One of the easiest things to accomplish is to make an effective loud speaker out of a gearcase. Large, flat surfaces, although the easiest to draw, should be avoided. When any great production of any particular unit is contemplated, such as a change-gear box for an automobile transmission, it would be wise to experiment with the shape of the case, after the other features of design had been settled, in order to see what could be accomplished in muffling the sounds inside by changes in the shape. Such changes in the shape would affect both the volume and the quality of the sounds transmitted. As stated before, the ideal case would be one that would not respond to any tone.

In addition to their own faults, gear teeth often reveal faults of other parts of the mechanism. Gear teeth form a loose connection that is very favorable to the transfer of vibration into sound. This sound, amplified by the gear case, often produces a result that is objectionable.

Sliding Gears.—It is essential that gears in operation rotate about a fixed center. This means that they should be firmly attached to the shaft that they drive. With sliding gears, this result cannot be attained, because, in order to slide freely, there must be sufficient looseness or play between the gears and shaft in order to prevent binding. In machine tool construction, sliding gears are usually mounted on shafts with one or two long keys or splines. In automobile construction, the shafts have several splines, the number usually ranging from six to twelve.

It is questionable whether a large number of splines has any beneficial result when used with sliding gears. It is not possible to attain a high enough degree of accuracy in the machining of the splined holes and shafts always to secure an equal bearing on all the splines. As a result, only a few of them drive. The effect may be to force the gear off center in order to equalize the pressure on three of these splines. If two of them should be adja-

cent splines, there is a possibility that the gear will shift its position during rotation and thus introduce further variations in the tooth action, with all the attendant noise. In many respects, it would seem as though a three-spline shaft would be more logical for sliding gears. In this case, the pressure against the splines would have a constant centering influence that should improve the operation of the gears. It is significant to note in this connection that the quietest transmission of which the writer has any knowledge employs a three-spline shaft for the sliding gears. This is another subject that would well repay a careful investigation.

Gears employed to drive parts of appreciable mass should not be rigidly attached to them but should have some degree of flexibility in the connection. This flexibility may be obtained by the use of a sufficiently long shaft or by means of a flexible coupling. This flexibility is desirable in order to prevent or to minimize both the effects of irregularities of the tooth action on the driven mass and the effects of variations in the load developed by the driven mass on the tooth action of the gears.

Flexible couplings are also used to compensate for any misalignment of the connected shafts. The primary purpose of a flexible coupling used on gear drives, however, is to absorb sudden shocks from whatever cause and to dampen out vibrations from both the gears and the driven mechanism.

LUBRICATION OF GEARS

[1]A certain amount of sliding takes place between the profiles of all gear-tooth forms in operation. This sliding creates friction, which can be partially overcome by lubrication. If no lubricant is introduced between the sliding surfaces, the friction is what may be termed "solid friction." If sufficient lubricant is always present, so that the two surfaces never touch each other, the friction is what may be termed "fluid friction." When lubricant is present between the sliding surfaces but the pressure is so high that the film of lubricant breaks down or for any other reason the sliding surfaces are not always separated by a film of lubricant, we have a condition of semilubricated surfaces.

All surfaces are more or less rough; even surfaces that are well machined and polished show under the microscope small projections and depressions. It is the interlocking of these minute

[1] Abstracted from "Practice of Lubrication"—Thompson.

projections that causes solid friction when two unlubricated surfaces are pressed together and move relative to one another. In general terms, the laws of solid friction are as follows:

The frictional resistance is (*a*) directly proportional to the total pressure between the surfaces; (*b*) independent of the rubbing speed of the surfaces at low speeds but decreases at very high speeds; (*c*) independent of the areas of the surfaces; (*d*) dependent to a considerable extent on the roughness and hardness of the surfaces.

These laws apply whether the motion is rolling or sliding. Owing to the surface irregularities, wear takes place, the softer material being more rapidly abraded than the harder. The wear and friction are much less for hard and smooth surfaces than for soft and rough surfaces. Surfaces of exactly the same materials are more inclined to seize and weld than dissimilar surfaces, hence, the reason why materials of different hardness and composition should usually be used for all rubbing surfaces. Two exceptions to this rule are the cases of cast iron on cast iron and hardened steel on hardened steel. In both examples, excellent results are obtained.

Although the friction between solid surfaces is independent of the area in contact, the wear is obviously the greater the smaller the area, because of the greater unit pressure.

By the introduction of a suitable third medium between the frictional surfaces, the solid friction may be partially or wholly eliminated. This medium may be solid, such as graphite, talc, white lead, and the like; semisolid, such as grease; or of an oily nature, such as lubricating oil.

Film Formation.—The object of all lubrication is that the lubricant should attach itself to the rubbing surfaces and form between them a film, which, under the conditions of speed, pressure, and temperature prevailing, will not be squeezed out but will keep the frictional surfaces apart. This object is not often attained, except in high-speed bearings.

In bearings thus perfectly lubricated, the "rubbing" surfaces never touch one another, and the friction is entirely dependent upon the lubricant. The laws governing fluid friction are totally different from the laws for solid friction and may be summarized as follows:

The frictional resistance (*a*) is independent of the pressure between the surfaces; (*b*) increases with the speed of the rubbing

surfaces; (*c*) increases with the area of the rubbing surfaces; (*d*) is independent of the condition of the rubbing surfaces or the material of which they are composed; (*e*) depends on the viscosity of the lubricant at the working temperature of the oil film.

The coefficient of friction for unlubricated surfaces ranges from 0.1 to 0.4; but with fluid friction, the coefficient of friction ranges from 0.002 to 0.010, according to the viscosity of the oil. It is therefore well worth while to approach as closely as possible a condition of fluid friction, because of both a saving in power and a reduction of wear.

Under conditions of low speed and high pressure, it is impossible, or at best extremely difficult, to obtain perfect film formation. With the great majority of gear drives, where the tooth action tends to wipe off the lubricant, where the unit pressures are very high, and where, at high speed, the lubricant is thrown off the teeth by centrifugal action, it is not possible to produce anything approaching perfect film formation. The surfaces, accordingly, are in an imperfectly lubricated or semilubricated condition, for which the coefficient of friction will range from 0.01 to 0.10, according to whether the surfaces are fairly well lubricated, thus approaching the condition of perfectly lubricated surfaces; or very poorly lubricated, thus approaching the condition of unlubricated surfaces.

There are no definite laws governing the friction of semilubricated surfaces. The frictional resistance is composed partly of solid friction and partly of fluid friction, and the more the solid friction predominates, the more important is the property known as "oiliness," and the less important is the viscosity of the lubricant. The object of lubrication of such surfaces is to make the best possible compromise between reduction of wear and reduction of fluid friction. For conditions of low pressure and high speed, the reduction of fluid friction is usually the most important point to consider, and its attainment demands low-viscosity oils of great oiliness; whereas, for conditions of high pressure and low speed, the reduction of wear must be given prime consideration, and its attainment therefore calls for viscous oils of great oiliness.

Lubricants.—Lubricants may be divided into three classes: first, lubricating oils; second, semisolid lubricants; and, third, solid lubricants. Semisolid lubricants are those that do not flow at ordinary room temperatures. Solid lubricants are those composed of solid materials, such as graphite, talc, soapstone, etc.

Some solid lubricants may be so finely divided as to enable them to be suspended in colloidal form in a liquid carrier. An example is "oildag," which is a diffusion of colloidal graphite and oil.

For efficient lubrication, oils must be selected to suit the operating conditions and the oiling system employed. In general terms, oils light in body should be employed for such conditions as high speeds, low pressures, low temperatures, and good mechanical conditions, while oils heavy in body should be employed for low speeds, high pressures, high temperatures, and bad or indifferent mechanical conditions.

All fixed oils (animal or vegetable) are more oily than mineral oils, and an admixture of a few per cent to the mineral oil will increase the latter's oiliness and assist in separating the rubbing surfaces more completely. If it were not for the high price of fixed oils and their tendency to gum (particularly the vegetable oils), they ought to be much more widely used than is the case at present.

Compounded oils also have the property of combining and emulsifying with water, so that their use is desirable where water may gain access to the gears. Water will displace a straight mineral oil and cause trouble but will combine with a compounded oil and form an emulsion or lather.

Semisolid or grease lubricants are advantageous in dusty and dirty surroundings, such as in cement mills, when the gears are not entirely inclosed. Grease should be used only where there are special reasons against the use of oil.

When a solid lubricant is introduced between otherwise unlubricated surfaces, the more or less finely divided particles of the lubricant associate themselves with one or other of the rubbing surfaces, filling in the pores and depressions. They act, as far as possible, as a smoothing and polishing agent and cover the original surfaces with a thin, smooth layer of the solid lubricant. As a result, the coefficient of friction is reduced, since the solid friction between the more or less rough original rubbing surfaces is replaced by the lesser solid friction between the smooth surfaces formed by the solid lubricant. When abrasion takes place, it occurs not so much between the original surfaces, which possess great cohesion, as between the particles of the solid lubricant, which have but little cohesion. If solid lubricants are employed, cutting and abrasion of the rubbing surfaces are therefore much less likely to occur.

There are a variety of conditions for which dry, solid lubricants have proved advantageous, as, for example, in such parts of machinery where the lubrication is apt to be neglected and which operate at low pressures and low speeds. When such surfaces are well coated with graphite, for example, and particularly if they are rubbed down to a dense, glazed finish, they will work upon each other for a long time, with comparative freedom and without cutting or wear's taking place.

Solid lubricants are also used in combination with other lubricants. The object of their use is (*a*) to reduce the solid friction; (*b*) to produce a smoothness of the rubbing surfaces that will assist in distributing the load evenly over all parts of the rubbing surfaces and thus allow a lower viscosity lubricant to be used and the fluid friction to be reduced; (*c*) to reduce the wear of the original surfaces and the risk of abrasion or cutting of the surfaces; (*d*) to reduce the consumption of lubricant.

When chains or gears are inclosed in an oil-tight casing, the use of oil is preferable to grease; in this case, the admixture of a small amount of finely pulverized graphite or colloidal graphite is often beneficial for the smooth operation of the gears.

Lubrication Systems.—The simplest method of lubricating the teeth of gears is to have the gear run in a bath of oil. This procedure requires an oil-tight case, and it is suitable for gears running at relatively low pitch-line velocities. As the speed of the gears increases, the excess oil carried around by the gear teeth is forced from between them at the time of meshing in a very short space of time, thus developing excess heat and also additional noise. This condition becomes more acute with increase of the faces of the gears as well as with increase of speed of the gears. Generally speaking, this method of lubrication should not be used when the pitch-line velocities of the gears exceed about 1,000 ft. per minute.

At the higher speeds, lubrication of the gear teeth may be accomplished by directing small streams of oil upon them. In general, the higher the speed, the finer should be such streams of oil. Because of the heating that develops when an excess of oil is present, high-speed gears often suffer from too much lubrication rather than too little. As a matter of fact, a large amount of oil that reaches the gear is thrown off against the case, thus creating a spray, so that very high-speed gears run in a bath or fog of oil mist. In many cases, this oil mist itself is sufficient to provide

lubrication for the gear teeth. For gears whose pitch-line velocities exceed 4,000 ft. per minute, effective lubrication of the teeth can usually be secured by directing a small stream of oil against the side of the gear blank.

For geared turbine drives, if a single oil system is used for both the turbine bearings and the gears, and if the oil gets mixed with water, the oil will suffer in the turbine system, to some extent; but when this same oil, mixed with minute particles of water and dirt, gets to the gears and is subjected to many times the bearing pressure, it is sure to suffer very quickly indeed. The result will be wear of the gearing. For these reasons, the oiling system for the gears should be made distinct and separate from the oiling system supplying the turbine bearings, quite apart from the question of whether the same oil or two different oils are used in the two systems. With separate oiling systems, the oil for the gears will remain dry and pure for a much longer time and will thus have a much better chance of keeping the teeth of the gears in good condition and preventing wear.

For automobile transmissions, the gears are lubricated most easily by means of oil, which should be filled into the gear box to the proper level. If the oil level is too high, the oil will run out of the gear shaft bearings, and excessive heat and noise will be developed. If the oil level is too low, the gears will not dip sufficiently to splash the oil to all parts requiring lubrication. It is obviously desirable to use as low a viscosity oil as possible, and the only reason why engine oil is not used is because the gear box is not sufficiently tight to permit the use of such an oil, so that, in order to prevent excessive leakage, a heavy-viscosity oil is used or even a semisolid lubricant, so-called "transmission greases." Transmission grease may be more economical than oil, but it has the disadvantage of causing greater loss in power, and it does not distribute itself effectively to all bearings, so that trouble is often experienced, particularly when there are ball or roller bearings in more or less inaccessible positions. About the only possible benefit that is secured in this location by the use of heavy oil or grease is its soft-pedal effect on the vibrations of the gear box and, hence, the noise of operation.

LUBRICATION OF ROLLING-MILL GEARS

From the viewpoint of lubrication, conditions in the rolling mill are different from those discussed heretofore. In fact, the

lubrication of blooming-mill gears, roll necks, and steam-engine cylinders involves probably more difficulties than any other phase of industrial plant operation.

The pinions adjacent to the roll necks in the blooming mill are usually inclosed in an oil-tight casing, in which they are run in a bath of specially prepared gear compound of high adhesive characteristics, having a viscosity of about 2,000 sec. Saybolt at 210° F. On the other hand, in some installations, these pinions may be covered with shields that are not oil-tight and are without bottoms. Bath lubrication, therefore, is out of the question, and the lubricant used must be able to stick tenaciously to the pinions over the periods that intervene between its applications and must maintain a sufficiently protective film. A viscosity of about 5,000 sec. Saybolt at 210° F. is necessary in order that the resultant lubricating film may be able to withstand the terrific pounding and hammering that occurs, especially when the mill is reversed.

Not only in the blooming mill, but also on all other rolls except in the plate and sheet mills, it is found necessary to run water constantly over the rolls and roll necks, for the dual purpose of cooling the rolls and blowing off the scale that may be formed as the ingots, bars, or billets are broken down. Such conditions, coupled with the extreme heat constantly encountered, place a most exacting requirement upon both the roll-neck and gear lubricants. These lubricants must be compounded products, inasmuch as straight mineral lubricants will not withstand the continued washing action of hot water. The usual procedure is to compound the gear lubricant with definite percentages of certain substances that will give the final product the desired adhesive properties. Any rolling-mill gearing, however, that does not come in contact with water can be lubricated with a straight mineral lubricant of a viscosity ranging from 2,000 to 5,000 sec. Saybolt, according to temperature conditions and the manner of lubrication.[1]

The frictional heat developed between the meshing teeth of gears spreads through the gears and shafts and into the bearings and gear case. Where the gears are not cooled by a circulating oil system, the whole of the heat developed must leave by radiation through the gear case into the atmosphere. The gears and their cases, therefore, assume a temperature higher than the surrounding room temperature, and the more the friction the

[1] Allen F. Brewer, of The Texas Company, in *Lubrication*, July, 1924.

greater will be the difference between the temperature of the gear unit and of the room. The difference is termed the "frictional rise in temperature," or simply the "frictional temperature," and forms a true guide as to the quality of oil in service and its method of application. Any reduction in the frictional temperature brought about by introducing another lubricant or changing its method of application will mean that this lubricant or method is better or more suitable for the conditions.

The frictional temperature remains practically constant for all room temperatures; that is, if the gear case temperature is 86° F. and the room temperature is 70°, the frictional temperature is 16°. If the room temperature rises to 74°, it will be found that the gear case temperature will rise to 90°; the friction developed is practically the same, and the gear case temperature must therefore be correspondingly higher, in order to radiate the same amount of heat into the atmosphere.

Oil as a Coolant.—When gears operate under conditions of high speed or pressure, the heat developed may become so great that it cannot be radiated from the surfaces of the casing with sufficient rapidity. Under such conditions, it becomes desirable or necessary to introduce a circulation oiling system by means of which the flow of oil that is directed on the gears not only serves to lubricate but also removes a large portion of the heat developed. This heat carried away with the oil can be radiated into the atmosphere from the oil tanks and pipes or, if necessary, can be removed by an oil-cooling arrangement.

If a greater flow of oil is required for purposes of cooling than is necessary for lubrication alone, the additional oil should be directed onto the sides of the gear blanks and not onto the teeth of the gears. Excessive oil on the teeth will develop further heat, because of its rapid expulsion from between the gear teeth at the meshing point. As noted before, a flow of oil on the sides of the gear blanks alone will usually provide sufficient lubrication for the teeth of high-speed gears.

The temperature of the gear case is an indication of the temperature of the gears, but it should be realized that the actual temperature of the gears will be appreciably higher than that of the case. If the frictional-temperature rise, as indicated by the temperature of the case or lubricating oil, is more than 50° F., steps should be taken either to improve the lubricating conditions or to introduce an oil circulating and cooling system.

Because of this frictional heat, high-speed gears must be made with sufficient backlash so that the teeth will not bind when the gears are heated and expanded.

As the actual amount of frictional heat developed by a pair of gears depends upon so many variables, such as the condition of tooth surfaces, speed, pressure, and the lubricating medium, it is impossible to calculate in advance the exact temperature rise that will be present. The best that can be done is to make an approximation. Such an approximation has its value, however, as in many cases it should make it possible to determine whether or not an oil circulation system should be provided. We must first make two assumptions: first, the amount of heat of friction to be dissipated; and second, the rate of dissipation from the case. The amount of heat of friction will be based upon the power transmitted by the gears and their assumed efficiency.

Well-made spur gears—and only such should be considered—show very high efficiencies. For our present purpose, if we assume a loss of power due to friction of 1 per cent of the load transmitted by the teeth, the approximation should be higher, if anything, than the actual conditions. The rate at which the heat will be dissipated is a very uncertain factor and will depend largely upon whether or not there is a free circulation of air around the gear case. Under the most favorable conditions in this respect, we will assume a rate of 2.7 B.t.u. per hour per square foot of exposed surface of the gear case as the rate at which heat is dissipated for a difference of 1° F. between the temperature of the gear case and the room temperature. This will be equivalent to about 35-ft.-lb. per minute per square foot of exposed surface. Thus, when

Q = difference between temperature of gear case and room temperature in degrees Fahrenheit.
W = load transmitted by gear teeth in foot-pounds per minute.
A = area of exposed surface of gear case in square feet.

$$Q = \frac{W}{3{,}500A} \qquad (72)$$

Under less favorable conditions, when the gear case is located in a corner or pit where there is but little free circulation of air, this rate at which heat will be dissipated will be materially less than the foregoing. Thus, with the gear case set in a pocket, the rate of dissipation of heat may be reduced to about 1.8 B.t.u.

per hour per square foot of exposed surface, which would be equivalent to about 24 ft.-lb. per minute per square foot of exposed surface. Under such conditions,

$$Q = \frac{W}{2{,}400A} \tag{73}$$

As an example, we will assume a reduction gear unit with a single pair of gears that transmits 100 hp. with a case which has an exposed area of 20 sq. ft. This would give us the following values:

$$W = 3{,}300{,}000$$
$$A = 20$$

If this gear case were situated in a favorable position as regards the free circulation of air around it, we would use Eq. (72).

$$Q = \frac{3{,}300{,}000}{3{,}500 \times 20} = 47^\circ \text{ F.}$$

Under these conditions it should not be necessary to provide an oil-cooling system. If this gear case were located in a pocket or other position where it would not be exposed to a free circulation of air, we would use Eq. (73).

$$Q = \frac{3{,}300{,}000}{2{,}400 \times 20} = 69^\circ \text{ F.}$$

Under these conditions, if the unit were to be used in continuous service, an oil-cooling system would probably be necessary. If it were to be used only for intermittent service, it would probably be possible to do without an oil-cooling system.

If a similar gear unit were involved which gave a double reduction through two pairs of gears, the value of W would be doubled because of the two gear meshes involved. This, in turn, would also double the value of Q.

CRITICAL SPEEDS

[1] It is a well-known fact that some high-speed gears have proved successful while others have not, though made with the same care and by the same concern. Sometimes gears show pitting on the tooth surfaces that cannot be explained by fatigue or by the amount of load transmitted. Some of these drives, furthermore, are noisy at their operating speed. These unsatisfactory condi-

[1] Abstracted from KENT, "Mechanical Engineers' Pocket Book."

tions seem to exist at times despite the care and attention given to the design and accuracy of the gear teeth.

We will therefore, for the present, take out attention from the gear teeth and direct it to the shafts and other rotating masses. It is possible that many failures may be due to excessive oscillations in the elastic system that comprises the whole unit.

The center of mass of a rotating body, such as a gear and its shaft, never lies in its mechanical center of rotation. Even if this mass is balanced perfectly, there is a deflection due to the weight of the rotating bodies as well as that caused by the tooth pressure of the gears. Thus, with a horizontal shaft, and with the direction of the gear-tooth pressure downward, the center of mass will lie below the true axis of rotation. When rotation begins, the shaft remains bent downward, but as the speed increases, the centrifugal force increases until the speed reaches a point known as the "critical speed," when the centrifugal force equals the force tending to bend the shaft downward, and excessive vibrations are set up. If the speed is increased above this critical speed, these vibrations disappear, and the revolving mass will rotate on an axis more nearly corresponding to its center of mass. Thus, the critical speed occurs at the time when the axis of rotation shifts from the mechanical center of rotation to a point closer to the actual center of mass. This transition point is the first or lowest critical speed. There is, in addition, a series of secondary critical speeds of higher value but with diminishing amplitudes of vibration.

If there should be other vibrations present, the frequency of which correspond or harmonize with the vibrations at the critical speed, the vibrations at the critical speed will be increased in amplitude until, if not checked in some manner, failure will result.

Critical speeds exist in gear drives, and it is safe to say that those drives that run at a speed close to a critical speed will be unsatisfactory in operation. When gear defects, such as noise, pitting, or rapid destruction, are primarily the result of running at such critical speeds, there is one sure remedy, that is, to proportion the shafts and other rotating masses so that the resulting critical speed is changed to one a safe amount from the desired operating speed.

It is possible to figure critical speeds, but their calculation is a long and complicated process. Where important and expensive gear drives are involved, however, such calculations should be

made in spite of the large amount of work involved. The subject of critical speeds is such an extensive one in itself that no attempt will be made to treat it further here, except to say that a rough approximation often used is given by the following:

$$\text{Critical speed in revolutions per minute} = \frac{188}{\sqrt{y}} \tag{74}$$

where y = maximum shaft deflection in inches.

Some gear drives may run at speeds above their first critical speed. In starting, these should pass quickly through these speeds, so that extreme vibrations will not be built up. The running speeds of gears should be at least 30 per cent above or below the critical speed. If the critical speed should figure out only slightly above the running speed, the shaft should be made larger and stiffer, thus raising the critical speed.

CHAPTER VII

GEAR-TOOTH LOADS

It is obvious that the first step toward proportioning the gear teeth is to determine the loads that these teeth must carry. These loads are dependent upon the nature and amount of work that must be performed. The analysis of gear-tooth loads involves many interesting problems in the resolution of forces into their components and also the composition of forces to determine their resultant.

Mechanics treats of the action of forces and their effect. A *force* is sometimes defined as any cause that tends to alter a body's state of rest or of uniform motion in a straight line. A force is measured by the masses set in motion and by the velocities imparted to them in a unit of time. The engineer's unit of force is the weight of 1 lb. In addition to force, there are two other fundamental quantities in mechanics that are used in combination to express various conditions. These quantities are distance, measured in feet or inches, and time, measured in hours, minutes, or seconds.

The term "force" is limited to the case in which some movement of masses is produced. If a pressure or pull is exerted upon a body not free to move, such an inactive pressure or pull is called a *stress*. A stress may be numerically specified either as a total stress or as a stress per unit of area. The total stresses may be measured as equal to the forces that produce them.

When a force acts upon a body and that body moves in the direction of the force, that force is said to do work, and the work done by it is measured by the product of the force and the space through which the body has moved in that direction. Work is usually expressed as foot-pounds. The work of 1 ft.-lb. is equivalent to lifting the weight of 1 lb. a distance of 1 ft. against gravity.

Power, in mechanics, is the performance of a given amount of work in a given time and is expressed as foot-pounds per second, foot-pounds per minute, etc. The horsepower is a unit of power

adopted for engineering purposes. One horsepower is equal to 33,000 ft.-lb. per minute, or 550 ft.-lb. per second. The kilowatt, used in electrical work, is equal to 1.34 hp.

The importance of a force not passing through a point is called the *moment* of that force round that point. It is equal to the amount of the force multiplied by the shortest distance from the point to the line of application of the force. The moment is measured in inch-pounds or foot-pounds, but it is necessary to observe the distinction between foot-pounds of statical moment and foot-pounds of work. In mechanics, the moment is usually measured in inch-pounds, while work is usually measured in foot-pounds. This practice helps to make the distinction more apparent.

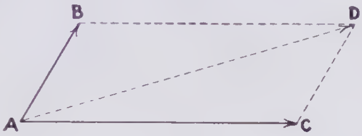


FIG. 83.—Composition of forces.

Forces have three characteristics: magnitude, direction, and point of application. When these three factors are given, the force is definitely established. Forces may be represented by straight lines; the length of the line indicating its magnitude, an arrowhead at the end of the line indicating its direction, and the position of the line indicating its point of application.

When two or more forces act simultaneously on a body, it will obey each as though the others did not exist, according to Newton's second law of motion. A single force that would produce the same effect upon a body as two or more forces acting together is called their *resultant*. The separate forces that can be so combined are called the *components*. The operation of finding the resultant is known as the *composition of forces*, while the operation of finding the components is known as the *resolution of forces*.

As an example, if the two forces represented in Fig. 83 by the lines AB and AC are applied at point A, we can assume that force AB acts first; and then AC, as represented by the dotted line BD, which is parallel to AC and equal to it in length. The diagonal line AB would then represent the resultant, or the single force that would have the same effect as the two single forces acting in different directions.

As a second example, if the four forces represented in Fig. 84 by the lines AB, AC, AD, and AE are applied at point A, we can

assume that force AB acts first; then AC as represented by the dotted line BF, which is parallel and equal in length to AC; then AD as represented by the dotted line FG; and then AE, which is represented by the dotted line GH. The resultant is represented by the dotted line AH.

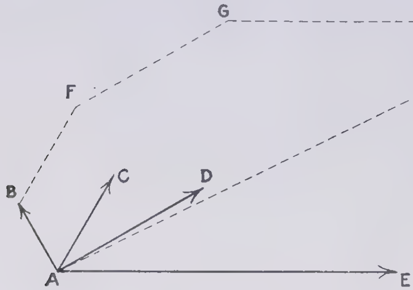


Fig. 84.

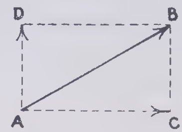


Fig. 85.

Fig. 84.—Finding the resultant of four forces graphically.
Fig. 85.—Two components of a single force.

As another example, if we resolve the force represented by the line AB in Fig. 85 into its components at right angles to each other, we draw the parallelogram as shown in dotted lines, with the line AB as its diagonal. The dotted lines AC and AD then represent the components of the force AB.

In the final analysis, the forces acting on or through gear teeth may be resolved into equivalent forces acting on simple levers. We will therefore consider briefly the moments of forces on a

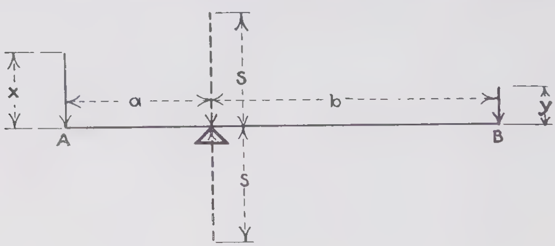


Fig. 86.—Forces acting on a lever at rest.

simple lever. Figure 86 is a diagram of a simple lever with a force x acting at A at a distance a from the fulcrum and a force y acting at B at a distance b from the fulcrum. If the effect of these forces is equal, their moments about the fulcrum must be equal. Hence, $ax = by$. If the force x is double the force y,

the distance a must be one-half the distance b, etc. Assuming a lever of no weight, the stress between the fulcrum and the lever is equal to the sum of the two forces. This stress exists in both the lever and the fulcrum, and the two stresses are equal and opposite, as represented by the dotted lines. When s represents this total stress, we have $s = x + y$.

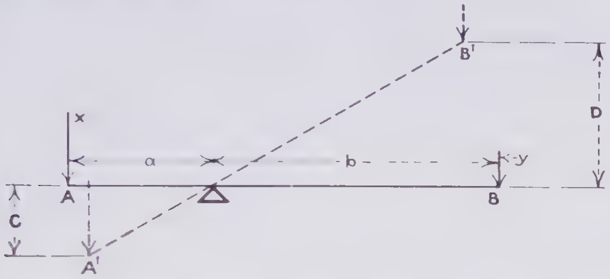


FIG. 87.—Work done by a lever in changing its position.

So long as the lever is stationary, no work is done. Figure 87 is a diagram of the lever in two positions, indicating movement. When the lever is moved, the point A has moved a vertical distance equal to C, and point B a vertical distance equal to D. These distances C and D are always proportional to the distances a and b, respectively. Ignoring friction, the work done is equivalent to lifting a weight corresponding to force y to a height of D or a weight corresponding to the force x to a height of C. Thus, the work done is equal to the product of the force and the distance through which it has acted. When a shaft is substituted for the fulcrum of the lever, and pulleys or gears are substituted for the lever, the foregoing conditions still hold true.

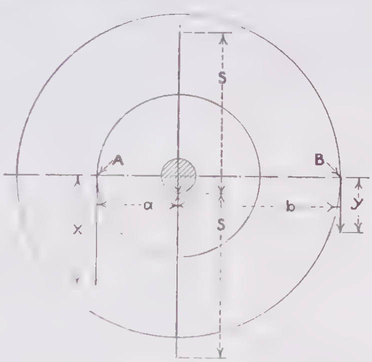


FIG. 88.—Analogy between a lever and a set of wheels.

In Fig. 88 we have the force x acting on the pulley A whose radius is equal to a and the force y acting on the pulley B whose radius is b. The product of the force x and the distance a is the moment of the force x in respect to the axis of the shaft and is also the torque on the shaft produced by the

force x. The torque is the moment of the force acting on a shaft in respect to its axis and is usually measured in inch-pounds.

A stress s as indicated on the diagram will exist between the shaft and its bearing. This stress is equal to the sum of the forces x and y and is known as the "bearing pressure." When the pulleys revolve, the work done is equal to the product of the force and the distance that the circumference of the pulley on which the force is acting has moved. The rate at which the work is done, or the power, is measured by the amount of work done in a given time. The more work that is done in a given period of time, the more the power that is employed to perform it; or, if the same amount of work is to be done in a shorter period of time, the more the power that will be required.

In essence, a pair of gears is but a combination of levers, the forces being applied through the teeth. Theoretically, these teeth are of such shape that the point of application of the force is always at the pitch point or the point where the two pitch circles are tangent to each other. The power transmitted by gears is always measured in terms of the tangential force at the pitch line, while involute gear teeth act at an angle to this tangent. It is therefore necessary to resolve this tangential force into its components in order to determine the actual loads on the gear teeth.

Figure 89 is a diagram of a pair of involute gears, where

r = pitch radius of pinion, inches
R = pitch radius of gear, inches
W = tangential force, pounds
P = pressure on gear teeth, pounds
α = pressure angle of gears
B = bearing pressure, pounds
A = force tending to separate axes of gears, pounds

From the right triangle representing the components of the force of which P is the hypotenuse and W is one leg, we have

$$P = \frac{W}{\cos \alpha} \text{ and } A = W \tan \alpha$$

In Fig. 89, the pinion is driving the gear in the direction shown by the arrows. If the gear were driving the pinion in the reverse direction to that shown, the diagram would remain unchanged.

But if the gear were driving the pinion in the direction shown, or if the pinion were driving the gear in the reverse direction, the diagram would also be reversed, but the magnitude of the force and its components would be unchanged.

The line representing the stress B or bearing pressure, as drawn in the diagram, shows the direction of pressure on the bearings. There would be equal and opposite stresses in the bearings themselves, but these have not been shown.

A change in the pressure angle of the gears will change the tooth pressures and bearing pressures, but this change is not so great as it is sometimes assumed to be. A change in pressure angle from 14½ to 25 deg. causes an increase in tooth and bearing pressure of less than 10 per cent. The greatest change is in the direction of the pressure. The bearing pressure is often confused with the force that tends to separate the axes of the gears. This is but one component, however, and has no importance by itself.

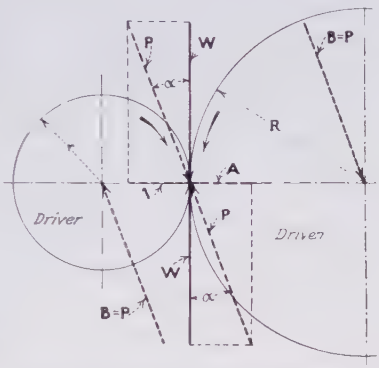


FIG. 89.—Graphic representation of tooth and bearing loads.

The torque on the pinion shaft is equal to Wr in.-lb., and the torque on the gear shaft is eqaul to WR in.-lb.

When a pair of gears is required to transmit a definite amount of power at a specified speed, the first step is to determine the pitch-line velocity of the gears. When

V = pitch-line velocity in feet per minute
r = pitch radius in inches

$$V = \frac{2\pi r \times \text{revolutions per minute}}{12} \tag{75}$$

The next step is to determine the tangential tooth load W at the pitch line. Usually, the power to be transmitted is given in horsepower. In such cases,

$$W = \frac{33{,}000 \text{ hp.}}{V} \tag{76}$$

Sometimes the load to be transmitted is given as the torque load in inch-pounds. In such cases, if the torque on the pinion shaft is given,

$$W = \frac{\text{inch-pounds torque}}{r} \tag{77}$$

The first two equations can be combined so that it is not necessary to calculate the pitch-line velocities independently. Usually, however, it is necessary to know the pitch-line velocities in order to select suitable working stresses for the material in the gears. These first two equations combined will give the following:

$$W = \frac{63{,}025 \text{ hp.}}{r \times \text{revolutions per minute}} \tag{78}$$

We will take, as a definite example, a pair of gears that are to transmit 10 hp. The pitch diameter of the pinion will be 4 in., and that of the gear will be 16 in. The pinion is to revolve at 2,000 r.p.m., and the pressure angle of the gears will be 20 deg. This gives us the following values:

$$\begin{aligned} r &= 2.000 \text{ in.} \\ R &= 8.000 \text{ in.} \\ \alpha &= 20 \text{ deg.} \\ \text{revolutions per minute} &= 2.000 \end{aligned}$$

Solving for the pitch-line velocity, we have

$$V = \frac{4\pi \times 2{,}000}{12} = 2{,}094 \text{ ft. per minute}$$

Solving for the tangential or transmitted load, we have

$$W = \frac{33{,}000 \times 10}{2{,}094} = 157.56 \text{ lb}$$

Torque on the pinion shaft $= 157.56 \times 2 = 315.12$ in.-lb.
Torque on the gear shaft $= 157.56 \times 8 = 1{,}260.48$ in.-lb.
Solving for the actual pressures on the gear teeth, we have

$$P = \frac{157.56}{0.93969} = 167.67 \text{ lb.}$$

The bearing pressures for both the gear and the pinion are equal to this tooth pressure of 167.67 lb.

Friction has been ignored in all of the foregoing. It is not necessary to consider the tooth friction except when long trains,

planetary or differential gear trains, or large amounts of power are involved.

Except for the bearing pressures, the same analysis as used for a single pair of gears is also used for simple trains. The bearing loads, however, must be analyzed individually, in order to determine the resultant when two or more gears mesh with another. A train of three gears with their centers in a straight line is shown in Fig. 90. In this example, the pinion a is driving the intermediate gear b in the direction shown by the arrow, and this intermediate gear drives the last gear c in the train. As gear b is an idler, the tooth loads between gears b and c will be the

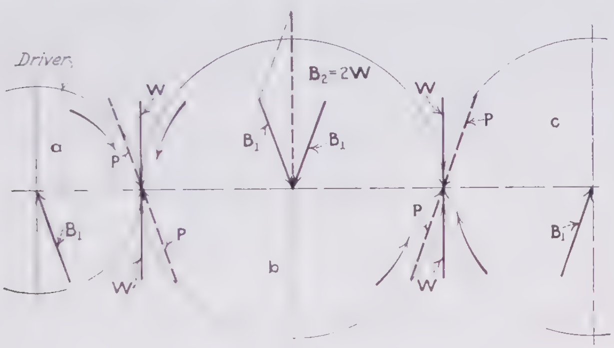


FIG. 90.—Typical gear-train problem.

same as those between a and b. The tooth pressure P is equal to $W \div \cos \alpha$. The bearing pressures for gears a and c are also equal to P, the same as in the case of a simple pair, as both of these gears have but a single mesh.

Bearing Loads.—The bearing pressure on the intermediate gear b is the resultant of the two bearing pressures that result from the two gear engagements. This resultant, as will be seen from Fig. 90, acts in a vertical direction down on the bearing and is equal to $2W$. This value of W would be determined as before. If, in this example, gears a and b were the same as those used as an example of a single pair, transmitting the same power at the same speed, the bearing pressure on the intermediate gear b would be equal to $2 \times 157.56 = 315.12$ lb. In effect, this intermediate gear is a lever loaded at both ends, so that the pressure on the fulcrum, or bearing, is equal to the sum of the two loads.

As another example, we will take the same train as shown in Fig. 90, but in this case, the intermediate gear b will transmit one-half the power it receives to its shaft and the remaining half to the third gear c. This will give the conditions shown in Fig. 91. The transmitted load W_2 between gears b and c will be one-half of the load W_1 transmitted from a to b. The bearing pressure for gear b will be the resultant of the two bearing pressures that result from the two gear engagements. If the load transmitted to gear c were less than half the original load, the lengths of the lines representing the forces would be correspondingly shorter. while if the load transmitted to gear c were more than half, the lengths of these lines would be correspondingly longer.

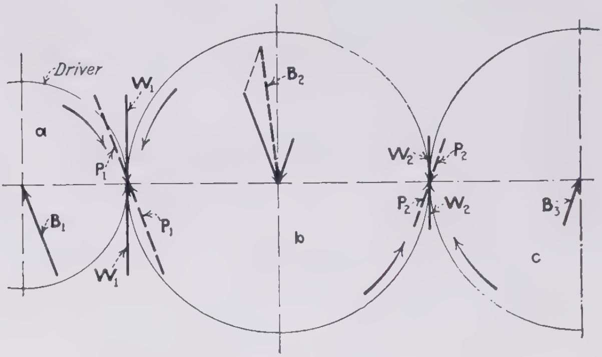


FIG. 91.—Typical gear-train problem.

As a further example of the analysis of the loads that are present with a train of gears, we will consider the case where the intermediate gear acts as an idler to drive two other gears, thus making a train of four gears. We will assume that one-half the power delivered to the intermediate gear is transmitted to each of the driven gears. In Fig. 92, this train is shown.

Gear a is driving in the direction shown by the arrow, transmitting the load W_1 to gear b. This load is divided equally between gears c and d. The lengths of the lines representing the forces W_3 and W_4 are proportional to the amount of power transmitted to the respective gears, and the sum of these two forces is equal to the original force W_1. The bearing pressure B_2 is the resultant of the three bearing loads B_1, B_3, and B_4 that result from the three gear engagements. In all cases, it is

necessary to make an individual analysis of the bearing pressure on any gear that meshes with more than one other gear, and the simplest way to make such an analysis is to make a layout, with the lengths of the lines representing the forces proportional to the magnitude of the forces, such as 1 in. equals 100 lb., or 1,000 lb., etc., as the case may be. The resultant of these forces can then be measured and converted into the load in pounds.

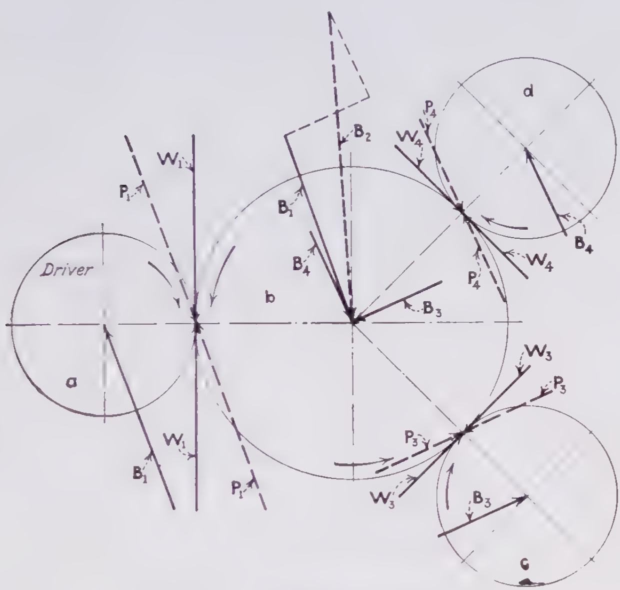


FIG. 92.—Typical gear-train problem.

Center Driving Gear.—We will now consider the conditions that exist when the driving gear is the middle one of three gears in a straight line, transmitting an equal amount of power to the two driven gears. In Fig. 93, such a combination is shown.

In this example, the bearing pressure on the driving gear *b* is the resultant of two equal and opposite pressures and is therefore equal to zero. If more power were transmitted to one gear than to the other, these two pressures would remain opposite but not equal. In such a case, the resultant would be the difference between the two pressures.

If the central driving gear were to drive three other gears equally spaced about its circumference, transmitting the same

amount of power to each of the driven gears, we would have the conditions shown in Fig. 94. In this example, the pressure on the bearing of the driving gear is the resultant of three equal pressures and is also equal to zero, as may be seen from their diagram in Fig. 94.

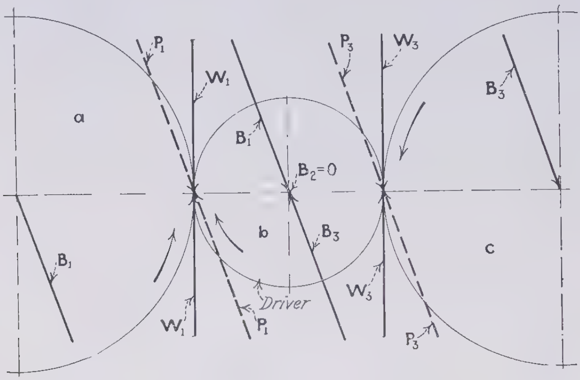


FIG. 93.—Typical gear-train problem.

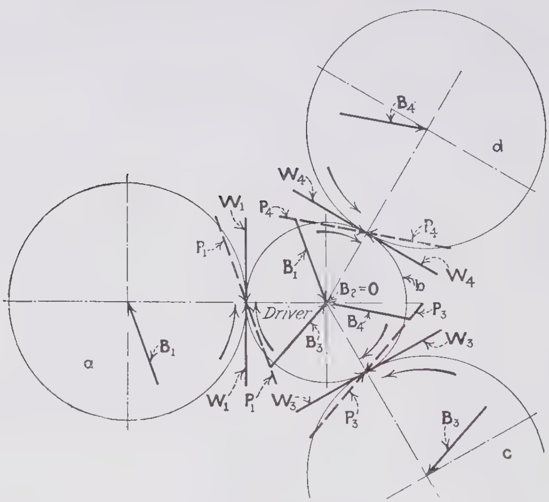


FIG. 94.—Typical gear-train problem.

The power transmitted by gears is the product of the tangential tooth load and the pitch-line velocity. Thus, in a reduction train of gears transmitting a fixed amount of power, the tooth loads will increase as the pitch-line velocities are reduced. The pitch-line velocities of two gears mounted on the same shaft and

revolving at the same rate of speed is directly proportional to their pitch diameters. Thus, the tooth loads on such gears are inversely proportional to their pitch diameters.

A train of reduction gears is shown in Fig. 95. If the pitch diameter of the intermediate pinion is one-half that of the intermediate gear, the tooth load on this pinion will be double the tooth load on the intermediate gear. The pressure on the bearings of the intermediate gear and pinion will be the resultant of the two pressures, as shown in the diagram.

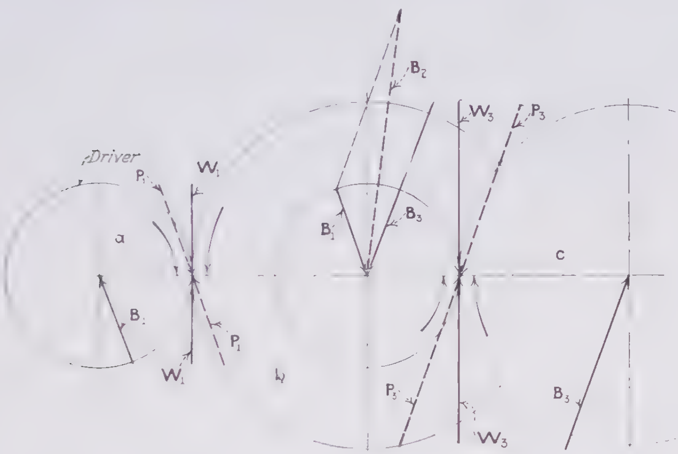


FIG. 95.—Typical gear-train problem.

Speeds and Loads in Planetary Gearing.—Thus far, we have been considering this subject, ignoring weight and friction. For most simple drives, such as single pairs and short trains, these factors can be safely ignored, because the weight of the gears themselves is but a small part of the total load and the friction losses between the gear teeth are but a very small part of the total friction loss in the mechanism. With well-made gears, these friction losses should not exceed about 1 per cent of the power transmitted at each mesh.

When the type of the gear drive departs from simple pairs and trains, however, very high tooth loads are often built up in combination with a much higher velocity of tooth engagement than the velocity at which the load is actually transmitted. In such cases, a careful analysis should be made, and the friction losses should be included. For purposes of such an analysis, we will

use a friction loss in the action of the gear teeth equal to 1 per cent of the tooth load at each gear mesh. This should represent a fair average under normal conditions of speed, method of lubrication, and load, such as are encountered in general machine practice.

In addition to the analysis of tooth loads, the determination of the relative speeds of the driving and driven gears becomes more complex as we depart from simple pairs and trains. We will therefore consider this phase of the subject also in addition to the tooth loads and friction power losses.

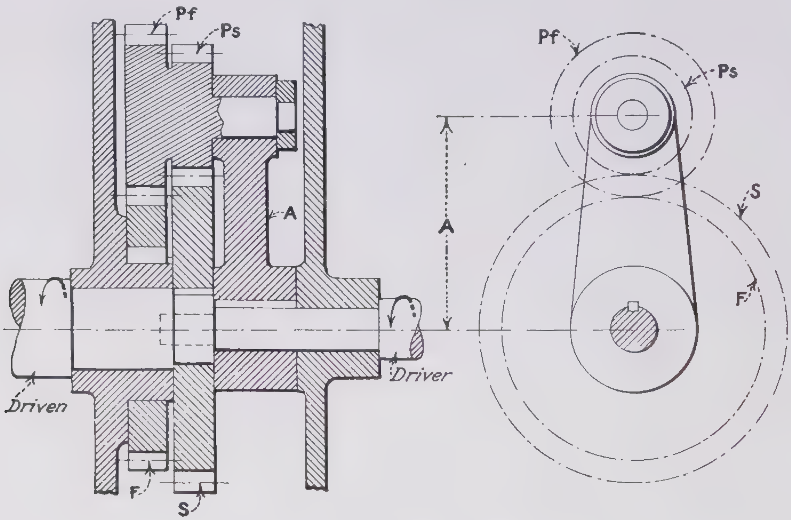


Fig. 96.—Typical planetary gear train.

One of the most complex types of gear trains to analyze is that known as "planetary gearing," or "epicyclic gearing," as it is also commonly called. Planetary gearing is composed of a train of gears where some of them revolve on fixed centers while others revolve on moving centers. Planetary gear trains may be arranged in a great variety of ways. In Fig. 96 is shown one arrangement of this type of gear train consisting of spur gears.

The gears that rotate on moving centers are known as "planet gears" or "pinions," while those that revolve on fixed centers are known as "sun gears." One of the sun gears is usually fixed. The planet pinions are carried on an arm that is keyed to the driving shaft, while the moving sun gear is keyed to the driven

shaft. The following notation will be used throughout in the analysis of planetary gear trains:

F = fixed sun gear
S = moving sun gear
Pf = planet pinion meshing with fixed sun gear
Ps = planet pinion meshing with moving sun gear
A = arm carrying planet pinions. In Fig. 96 it is the driving member.

When these symbols are used in equations, they will represent the pitch diameters of the gears in inches, and a will represent the center distance between the sun gears and planet pinions in inches.

We will first determine the reduction ratio of the train shown in Fig. 96. The simplest method on any planetary train is to consider all of the gears in the train as locked together and the entire combination revolved for one turn. As one member of the train is fixed, this member F must then be revolved back one turn while the driving arm A is held fixed. The arm A has made one revolution while the moving sun gear S has made one revolution plus or minus the amount it moves when the fixed sun gear is returned to its original position. This amount is determined just as if the gears were a simple train and is the amount the moving sun gear revolves for each revolution of the arm A.

In the example shown in Fig. 96, when all of the gears in the train are locked together and revolved for one turn, the driven shaft will make one turn in the direction of the driver. Then, when the arm A is fixed, and the sun gear F is revolved for one turn in the reverse direction, the sun gear S will also be turned backward an amount depending upon the ratio in the gear train between F and S considered as a simple train. This amount is equal to $\dfrac{F \times Ps}{Pf \times S}$

$$\text{whence, reduction ratio} = 1 - \frac{F \times Ps}{Pf \times S}$$

When the result obtained from this equation is plus, the driven member will revolve in the same direction as that of the driving member. When the result is minus, the driven member will revolve in the opposite direction to that of the driving member.

In the construction shown in Fig. 96, when F is smaller than S, the final drive will be in the same direction as the original one; and when F is larger than S, the driven shaft will rotate in the opposite direction to the driver.

As a definite example, we will determine the reduction ratio of a planetary gear train, as shown in Fig. 96, with the following values:

$$F = 11.000 \text{ in.}$$
$$S = 12.000 \text{ in.}$$
$$Pf = 5.000 \text{ in.}$$
$$Ps = 4.000 \text{ in.}$$
$$A = 8.000 \text{ in.}$$

$$\text{Reduction ratio} = 1 - \frac{F \times Ps}{Pf \times S} = 1 - \frac{11 \times 4}{5 \times 12} = +\frac{4}{15}$$

If we reversed these values so that F is the larger sun gear, we should have the following

$$F = 12.000 \text{ in.}$$
$$S = 11.000 \text{ in.}$$
$$Pf = 4.000 \text{ in.}$$
$$Ps = 5.000 \text{ in.}$$
$$a = 8.000 \text{ in.}$$

$$\text{Reduction ratio} = 1 - \frac{F \times Ps}{Pf \times S} = 1 - \frac{12 \times 5}{4 \times 11} = -\frac{4}{11}$$

We will now consider the tooth loads that exist with a planetary train of gears, as shown in Fig. 96. As our first example, we will take a train where F is smaller than S. In Fig. 97, a diagram of such a train is shown.

We will consider only the extent of the tangential tooth loads on these planetary trains. The bearing pressures would be determined from these tooth loads in exactly the same manner as for simple trains.

In effect, in the planetary gear train shown in Fig. 97, the pitch point on the fixed sun gear F is the fulcrum of a lever. The driving force is acting on the center of the planet pinions to pry the moving sun gear S ahead. At the right, in Fig. 97, is shown the diagram of a simple lever that represents these conditions. The force W_1 is the driving force, and W_2 is the tangential tooth load on the driven sun gear S, while the difference between them is the tooth load on the fixed sun gear F, because force W_1 acts in the opposite direction to force W_2, hence, W_3 is equal to their

difference. We know, from a simple lever, that $W_1 \times b = W_2 \times c$ and $W_3 = W_2 - W_1$. We have, from the diagram,

$$c = \frac{S - F}{2} \text{ and } b = \frac{2A - F}{2}$$

$$W_2 = \frac{b}{c} W_1 = \frac{2A - F}{S - F} W_1 \text{ and } W_3 = W_1\left(\frac{2A - F}{S - F} - 1\right)$$

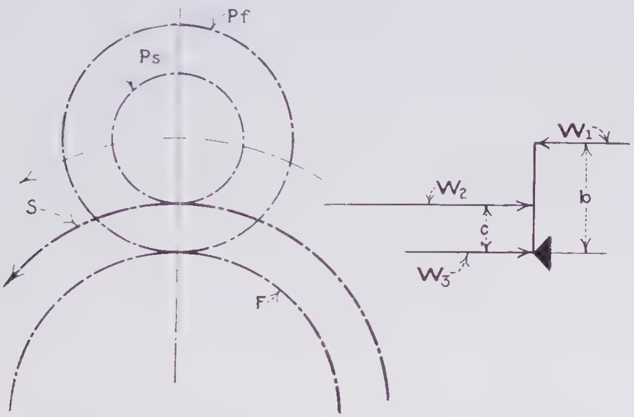


FIG. 97.—Lever analogy applied to planetary gear train.

When we know the amount of power to be transmitted and the speed of either the driving or driven shaft, we can determine the magnitude of these loads. Thus, when

hp. = horsepower
r.p.m. = revolutions per minute of driving shaft
V = velocity in feet per minute of point of application of driving force

$$V = \frac{2\pi a \times \text{r.p.m.}}{12} = 0.5236\,a \times \text{r.p.m.}$$

$$W_1 = \frac{33{,}000 \text{ hp.}}{V} = \frac{63{,}025 \text{ hp.}}{a \times \text{r.p.m.}}$$

As a definite example, we will assume that a planetary gear train with the following values is to transmit 2 hp. when the driving shaft revolves 1,200 r.p.m.

F = 11.000 in.
S = 12. 000 in.
Pf = 5.000 in.
Ps = 4.000 in.
a = 8.000 in.

We have already seen that the reduction ratio for such a train is $+4/15$. In this example, the driven shaft will revolve $4/15 \times 1{,}200 = 320$ r.p.m. in the same direction as the driving shaft. We therefore have, for the loads,

$$W_1 = \frac{63{,}025 \times 2}{8 \times 1{,}200} = 13.13 \text{ lb.}$$

$$W_2 = 13.13\left(\frac{16 - 11}{12 - 11}\right) = 65.65 \text{ lb.}$$

$$W_3 = 13.13\left(\frac{16 - 11}{12 - 11} - 1\right) = 52.52 \text{ lb.}$$

As a check on these loads, we know that the sun gear S revolves at 320 r.p.m. and transmits 2 hp. Thus, we have

$$W_2 = \frac{63{,}025 \times 2}{6 \times 320} = 65.65 \text{ lb.}$$

The pitch-line velocity of the sun gear S about its own axis is equal to $0.5236 \times 6 \times 320 = 1{,}005$ ft. per minute. The speed of the tooth engagements that determine the power loss due to friction of the gear teeth is much higher than this, however. This pitch-line velocity of engagement, which will be called Ve, will be equal to the pitch-line velocity of the planet pinion Ps in relation to any fixed point on the arm A.

This pitch-line velocity of the planet pinion Ps is controlled by that of the planet pinion Pf, which meshes with the fixed sun gear F. These pitch-line velocities of the two planet pinions will be directly proportional to their diameters. The pitch-line velocity of the planet pinion Pf is equal to the product of the pitch-circle circumference of the fixed sun gear F and the speed of the arm A. Whence, the pitch-line velocity of engagement of the planet pinion Pf is equal to

$$\frac{\pi F \times \text{r.p.m.}}{12} = \frac{3.1416 \times 11 \times 1{,}200}{12} = 3{,}456 \text{ ft. per minute.}$$

The pitch line velocity of engagement Ve, of the planet pinion Ps, is equal to $3{,}456 \times \dfrac{Ps}{Pf} = 2{,}765$ ft. per minute. With a pair of gears operating on fixed centers at this pitch-line speed with the tooth load of 65.65 lb., these gears would be transmitting $65.65 \times 2{,}765 = 181{,}522$ ft.-lb. per minute. This result is what we may call the "potential work accomplished." The

actual work is 2 hp., or 66,000 ft.-lb. per minute. In this example, the potential work is nearly three times as much as the actual. Assuming 1 per cent power loss due to tooth friction, we would lose at this mesh 1,815 ft.-lb. per minute. In order to deliver 2 hp. to the driven shaft, this additional power must be transmitted through the first pair of gears.

At this first pair, also, the potential power is greater than the actual. V_e here is 3,456 ft. per minute, while the tooth load is 52.52 lb., whence the potential power transmitted is equal to $52.52 \times 3{,}456 = 186{,}909$ ft.-lb. per minute + 1,815 ft.-lb. per minute making a total of 188,724 ft.-lb. per minute. Using the same factor of 1 per cent as before for the power loss due to tooth friction, we have a power loss in the first pair of gears of 1,887 ft.-lb. per minute. This, added to the loss in the second pair, makes a total loss of about 3,702 ft.-lb. per minute, or about 0.11 hp., when transmitting 2 hp. to the driven shaft.

Furthermore, in order to carry this tooth load at the speed developed for tooth engagements, both pairs of gears must be of such a size that they would transmit about 6 hp. safely as a simple pair on fixed centers. In other words, with a reduction of about 4:1, the potential power that must be transmitted by the gear teeth is about three times the actual power transmitted. A single pair of gears that would give the same speed reduction would have a power loss due to tooth friction of about 0.02 hp.

In design, this construction of planetary gear trains will give large variations in speed in either direction within a very small compass. The large amount of potential power that must be handled together with the resulting power losses make such trains very inefficient and also much bulkier than might seem at first glance, if they are used to transmit any great amount of power with any large variations in speed. These trains have their chief value for use where the loads are intermittent and power losses are of minor importance, such as in chain hoists, and also where the loads are very small and a large reduction is required in a minimum space.

As the speed ratio is increased, the power losses also increase. As a second example, we will determine the tooth loads and power losses on a similar planetary train that gives a reduction of about 30:1 instead of 15:4 and is to transmit 2 hp. at 1,200 r.p.m. of the driving shaft. Such a train might have the following values:

$$F = 12.000 \text{ in.}$$
$$S = 12.100 \text{ in.}$$
$$Pf = 4.000 \text{ in.}$$
$$Ps = 3.900 \text{ in.}$$
$$a = 8.000 \text{ in.}$$

$$\text{Reduction ratio} = 1 - \frac{F \times Ps}{Pf \times S} = 1 - \frac{12.0 \times 3.9}{4.0 \times 12.1} = +\frac{4}{121}$$

Referring again to Fig. 97,

$$W_1 = \frac{63.025 \text{ hp.}}{a \times \text{r.p.m.}} = 13.13 \text{ lb.}$$

$$W_2 = W_1\left(\frac{2a - F}{S - F}\right) = 13.13\left(\frac{16 - 12}{12.1 - 12}\right)525.20 \text{ lb.}$$

$$W_3 = W_2 - W_1 = 525.20 - 13.13 = 512.07 \text{ lb.}$$

As a check on these loads, the driven gear S revolves $1{,}200 \times \frac{4}{121} = 39.67$ r.p.m., whence,

$$W_2 = \frac{63{,}025 \times 2}{6.05 \times 39.67} = 525.20 \text{ lb.}$$

The pitch-line velocity of engagement Ve of the planet pinion Pf is equal to $0.5236 + 6 \times 1{,}200 = 3{,}770$ ft. per minute.

The pitch-line velocity Ve of the planet pinion Ps is equal to $3{,}770 \times \frac{Ps}{Pf} = 3{,}676$ ft. per minute.

With a tooth load W_2 of 525.20 lb., the potential power transmitted by the second pair is equal to $525.20 \times 3{,}676 = 1{,}930{,}635$ ft.-lb. per minute. A power loss of 1 per cent at this mesh is equal to 19,306 ft.-lb. per minute.

With a tooth load W_3 of 512.07 lb., the potential power transmitted by the first pair is equal to $512.07 \times 3{,}770 = 1{,}930{,}504$ ft.-lb. per minute + 19,306 ft.-lb. per minute, making a total of 1,949,810 ft.-lb. per minute. A power loss of 1 per cent for this pair would amount to 19,498 ft.-lb. per minute, making a total power loss due to tooth friction on this train of 38,804 ft.-lb. per minute, or about 1.18 hp., when transmitting but 2 hp. In other words, the power input must be 3.18 hp. to deliver 2 hp. to the driven shaft. A simple reduction train of three pairs would have a corresponding power loss of about 0.06 hp.

In this example, the potential power transmitted through each pair is about 20 times the actual power with a reduction ratio of about 30:1. As the reduction ratio increases, the potential power

transmitted will have about the same ratio to the actual power as the reduction ratio. Thus, in a planetary gear train of this construction, with a reduction of 100:1, the potential power transmitted through each pair, will be about 100 times the actual power. With a 1 per cent power loss when transmitting 2 hp., the friction loss on the gear teeth alone would amount to about 4 hp., thus requiring an input of 6 hp. to deliver 2 hp.

We will now examine the conditions that exist in a planetary reduction train when the sun gear F is larger than the sun gear S, so that the power is delivered in the opposite direction of rotation. In Fig. 98, a diagram of such a train is shown.

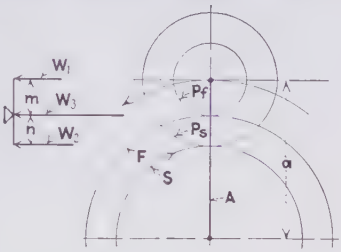


FIG. 98.—Analysis of external planetary gear train.

We already know that the reduction ratio for this type of planetary gear train is equal to $\left(1 - \frac{F \times Ps}{Pf \times S}\right)$. In this case, with F larger than S, the solution will give a minus value, which means that the driven gear S will revolve in the opposite direction to the driving member A.

As before, the action between these gears can be compared to a simple lever. In effect, the pitch point on the fixed sun gear F is the fulcrum, while the load is applied through the center of the planet pinions to force the moving sun gear S backward. At the left, in Fig. 98, is shown a diagram of a simple lever that represents these conditions. The force W_1 is the driving force, W_2 is the tangential tooth load on the driven sun gear S, while W_3 is the pressure against the fulcrum at the pitch point of the fixed sun gear F. As forces W_1 and W_2 are acting in the same direction, the pressure on the fulcrum W_3 is equal to their sum. We know that $W_1 \times m = W_2 \times n$ and $W_3 = W_1 + W_2$. We have, from the diagram,

$$n = \frac{F - S}{2} \quad \text{and} \quad m = \frac{2a - F}{2}$$

$$W_2 = \frac{m}{n} W_1 = W_1\left(\frac{2a - F}{F - S}\right)$$

When we know the amount of power to be transmitted and the speed, we can readily determine the magnitude of the loads

When V = velocity in feet per minute of point of application of driving force,

$$V = \frac{2\pi a \times \text{r.p.m.}}{12} = 0.5236a \times \text{r.p.m.}$$

$$W_1 = \frac{33{,}000\ \text{hp.}}{a \times V} = \frac{63.025\ \text{hp.}}{a \times \text{r.p.m.}}$$

As a definite example, we will assume that a planetary gear train with the following values is to transmit 2 hp. when the driving shaft revolves 1,200 r.p.m.:

$$\begin{aligned} F &= 12.100\text{ in.} \\ S &= 12.000\text{ in.} \\ Pf &= 3.900\text{ in.} \\ Ps &= 4.000\text{ in.} \\ a &= 8.000\text{ in.} \end{aligned}$$

$$\text{Reduction ratio} = 1 - \frac{F \times Ps}{Pf \times S} = 1 - \frac{12.10 \times 4.00}{3.90 \times 12.00} = -\frac{4}{117}$$

The driven shaft revolves, therefore, $1{,}200 \times 4/117 = 41$ r.p.m. in the opposite direction to that of the driving shaft. We have, for the loads,

$$W_1 = \frac{63{,}025\ \text{hp.}}{a \times \text{r.p.m.}} = \frac{63{,}025 \times 2}{8 \times 1{,}200} = 13.13\text{ lb.}$$

$$W_2 = W_1\left(\frac{2a - F}{F - S}\right) = 13.13\left(\frac{16.00 - 12.10}{12.10 - 12.00}\right) = 512.07\text{ lb.}$$

$$W_3 = W_2 + W_1 = 525.20\text{ lb.}$$

The pitch-line velocity of the sun gear S about its own axis is equal to $0.5236 \times 6.0 \times 41 = 129$ ft. per minute. The pitch-line velocity of tooth engagement Ve that controls the power loss due to tooth friction is much higher than this and must be determined in the same manner as before.

The pitch-line velocity Ve of the planet pinion Pf is equal to $0.5236 \times 6.05 \times 1{,}200 = 3{,}801$ ft. per minute. The corresponding velocity of the planet pinion Ps is equal to $3{,}801 \times \frac{Ps}{Pf} = 3{,}899$ ft. per minute.

With a tooth load W_2 of 512.07 lb., the potential power transmitted by the second pair of gears is equal to $512.07 \times 3{,}899 = 1{,}996{,}561$ ft.-lb. per minute. A power loss of 1 per cent here is equal to 19,966 ft.-lb. per minute.

The potential power transmitted by the sun gear F and its mating planet pinion is equal to $525.20 \times 3{,}801 = 1{,}996{,}285$ ft.-lb. per minute + 19,966 ft.-lb. per minute. The total is, therefore, 2,016,251 ft.-lb. per minute. A 1 per cent power loss here is equal to 20,163 ft.-lb. per minute.

The total power loss due to gear-tooth friction alone would amount to 40,139 ft.-lb. per minute, or about 1.22 hp. when transmitting but 2 hp. A simple reduction train of three pairs giving the same reduction would have a power loss of about 0.06 hp. In this case, also, the potential power transmitted by the gears in the planetary train is about thirty times as great as the actual power with a speed reduction of about 30:1.

In the foregoing examples of planetary gear trains, the arm A has been the driving member, and the trains gave a reduction in speed to the driven member. If the arm A should be the driven member, with the sun gear S as the driving member, the speed ratio between the driver and driven would be unchanged, but the train would act as a speed-increasing unit instead of a reduction unit. When transmitting the same power at the same speeds, the tooth loads and power losses would be practically the same, regardless of whether the arm A was the driving or the driven member.

In these examples, only one pair of planetary pinions was used. Often two or more are used to reduce the unit pressures on the gear teeth and also to reduce the bearing loads. Thus, when a double set of planet pinions are used, the unit pressure, or pressure on the teeth of each pinion, is reduced to one-half of the total transmitted load, and the bearing pressures on the sun gear S and the main bearing of the arm A may be reduced to zero, but the total loads and power losses will remain unchanged.

Another type of planetary gear train is one that employs two internal gears as sun gears. In Fig. 99, a planetary gear train of this type is shown.

We will first determine the reduction ratio of such a train, in the same manner as before. In the construction shown, the arm A is the driving member and the internal sun gear S is the driven member. When all of the gears are locked together and revolved for one turn, both of the sun gears and the driven shaft will be advanced one turn. Then, when the arm A is held stationary and the fixed sun gear F is revolved backward one turn to restore it to its original position, the sun gear S will also turn backward

an amount depending upon the ratio in the gear train between F and S considered as a simple train. This amount is equal to $\frac{F \times Ps}{Pf \times S}$, whence,

$$\text{Reduction ratio} = 1 - \frac{F \times Ps}{Pf \times S}$$

This equation is the same as when two spur gears are used as sun gears. As before, when the solution is plus, the driven member will revolve in the same direction as the driving member; when the solution is minus, the driven member will revolve in the opposite direction to the driving member. In this construction, when the fixed sun gear F is smaller than the sun gear S, the final drive will be in the opposite direction to the original one, and when F is larger than S, both shafts will revolve in the same direction.

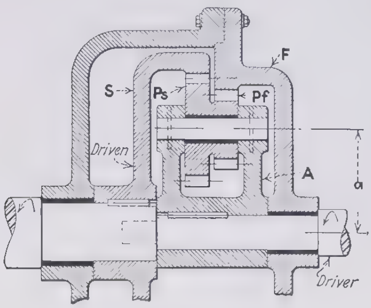


FIG. 99.—Internal planetary gear train.

Here, also, when the arm A is the driven member, the speed ratio of the planetary train will remain unchanged, but the unit will be a speed-increasing unit instead of a speed-reducing unit. The tooth loads and the power losses will also be approximately the same, regardless of whether the arm A is the driving or the driven member.

As a definite example, we will determine the reduction ratio of a planetary gear train, as shown in Fig. 99, with the following values:

$$\begin{aligned} F &= 11.000 \text{ in.} \\ S &= 12.000 \text{ in.} \\ Pf &= 3.000 \text{ in.} \\ Ps &= 4.000 \text{ in.} \\ a &= 4.000 \text{ in.} \end{aligned}$$

$$\text{Reduction ratio} = 1 - \frac{F \times Ps}{Pf \times S} = 1 - \frac{11 \times 4}{3 \times 12} = -\frac{2}{9}$$

If we reversed these values so that the fixed sun gear F is larger than the sun gear S, we should have the following:

$$F = 12.000 \text{ in.}$$
$$S = 11.000 \text{ in.}$$
$$Pf = 4.000 \text{ in.}$$
$$Ps = 3.000 \text{ in.}$$
$$a = 4.000 \text{ in.}$$

$$\text{Reduction ratio} = 1 - \frac{12 \times 3}{4 \times 11} = +\frac{2}{11}$$

We will now consider the tooth loads on a planetary train of gears, as shown in Fig. 99. As the first example, we will take a train where the fixed sun gear F is smaller than the sun gear S. In Fig. 100, a diagram of such a train is shown.

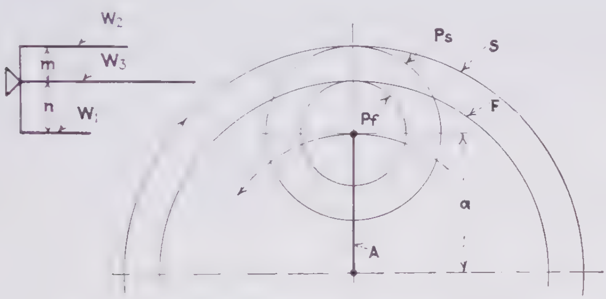


FIG. 100.—Analysis of internal planetary gear train.

As before, the action between these gears can be compared to a simple lever. In effect, the pitch point on the fixed sun gear F is the fulcrum, while the load is applied through the center of the planet pinions to force the sun gear S backward. At the left, in Fig. 100, is shown a diagram that represents these conditions. The force W_1 is the driving force, W_2 is the tangential tooth load on the driven sun gear S, while W_3 is the pressure against the fulcrum at the pitch point of the fixed sun gear F. As forces W_1 and W_2 are acting in the same direction, the pressure on the fulcrum, or W_3, is equal to their sum. We know from the diagram that $W_1 \times n = W_2 \times m$ and $W_3 = W_1 + W_2$. We also have, from the diagram,

$$m = \frac{S - F}{2} \quad \text{and} \quad n = \left(\frac{F - 2a}{2}\right)$$

$$W_2 = \frac{n}{m} W_1 = W_1 \frac{F - 2a}{(S - F)}$$

When the power and speed requirements are known, these loads may be readily determined. When

V = velocity in feet per minute of point of application of driving force

$$V = \frac{2\pi a \times \text{r.p.m.}}{12} = 0.5236 \times a \times \text{r.p.m.}$$

$$W_1 = \frac{33{,}000 \times \text{hp.}}{V} = \frac{63{,}025 \times \text{hp.}}{a \times \text{r.p.m.}}$$

As a definite example, we will assume that a planetary gear train with the following values is to transmit 2 hp. when the driving shaft revolves 1,200 r.p.m.

$$\begin{aligned} F &= 12.000 \text{ in.} \\ S &= 12.200 \text{ in.} \\ Pf &= 3.800 \text{ in.} \\ Ps &= 4.000 \text{ in.} \\ a &= 4.100 \text{ in.} \end{aligned}$$

$$\text{Reduction ratio} = 1 - \frac{F \times Ps}{Pf \times S} = \frac{12.00 \times 4.00}{12.20 \times 3.80} = -\frac{41}{1{,}159}$$

The driven shaft will revolve, therefore, $1{,}200 \times 41/1{,}159 = 42.45$ r.p.m. in the opposite direction to the driving shaft.

$$W_1 = \frac{63{,}025 \times \text{hp.}}{a \times \text{r.p.m.}} = \frac{63{,}025 \times 2}{4.10 \times 1{,}200} = 25.62 \text{ lb.}$$

$$W_2 = W_1\left(\frac{F - 2a}{S - F}\right) = 25.62\left(\frac{12.00 - 8.20}{12.20 - 12.00}\right) = 486.78 \text{ lb.}$$

$$W_3 = W_1 + W_2 = 25.62 + 486.78 = 512.40 \text{ lb.}$$

The pitch-line velocity of the sun gear S about its own axis is equal to $0.5236 \times 6.10 \times 42.45 = 136$ ft. per minute.

The pitch-line velocity of engagement of the planet pinion Pf is equal to $0.5236 \times 6.00 \times 1{,}200 = 3{,}770$ ft. per minute.

The corresponding pitch-line velocity of the planet pinion Ps is equal to $3{,}770 \times \dfrac{Ps}{Pf} = 3{,}968$ ft. per minute.

With a tooth load W_2 of 486.78 lb., the potential power transmitted between the sun gear S and its planet pinion is equal to $486.78 \times 3{,}968 = 1{,}931{,}543$ ft.-lb. per minute. A power loss of 1 per cent due to tooth friction would amount to 19,315 ft.-lb. per minute.

The pitch-line velocity of tooth engagement between the fixed sun gear F and its mating planet pinion Pf is equal to 3,770 ft. per minute, while the tooth load W_3 is equal to 512.40 lb. The potential power transmitted by this pair is equal to 512.20 × 3,770 = 1,931,748 ft.-lb. per minute + 19,315 ft.-lb. per minute. The total potential power transmitted by this pair is, therefore, equal to 1,951,063 ft.-lb. per minute. A power loss of 1 per cent here is equal to 19,511 ft.-lb. per minute, and the total power loss due to tooth friction alone on this planetary train is equal to 38,826 ft.-lb. per minute, or about 1.18 hp. when transmitting only 2 hp. In other words, it would require an input of about 3.2 hp. to deliver 2 hp. through this train. The power losses in

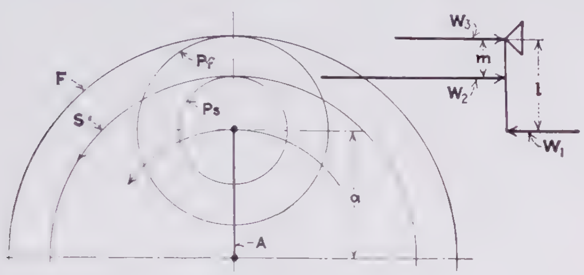


FIG. 101.—Analysis of internal planetary gear train, where F is larger than S.

this type of planetary reduction train are practically the same as for the first type studied.

As a second example of this type of planetary gear train, we will take one where the fixed sun gear F is larger than the moving sun gear S. In Fig. 101, a diagram of such a train is shown.

We already know that the reduction ratio for this type of planetary gear train is equal to $\left(1 - \frac{F \times Ps}{Pf \times S}\right)$. In this example, with F larger than S, the solution will be plus, which means that the driven gear S will revolve in the same direction as the driving arm A.

As before, the action between these gears can be compared to a simple lever. At the right, Fig. 101, is shown a diagram of a simple lever that represents the tangential-load conditions. The force W_1 is the driving force, W_2 is the tangential tooth load on the sun gear S, while W_3 is the tangential load on the teeth

of the fixed sun gear F. Whence, $W_1 \times l = W_2 \times m$ and $W_3 = W_2 - W_1$. We have, from the diagram,

$$m = \frac{F - S}{2} \quad \text{and} \quad l = \frac{F - 2A}{2}$$

$$W_2 = \frac{l}{m} W_1 = W_1\left(\frac{F - 2a}{F - S}\right)$$

As a definite example, we will take the following planetary gear train and determine the tooth loads and power losses when the arm A drives at 1,200 r.p.m., transmitting 2 hp.:

$$\begin{aligned} S &= 12.000 \text{ in.} \\ F &= 12.200 \text{ in.} \\ Ps &= 3.800 \text{ in.} \\ Pf &= 4.000 \text{ in.} \\ a &= 4.100 \text{ in.} \end{aligned}$$

$$\text{Reduction ratio} = 1 - \frac{F \times Ps}{Pf \times S} = 1 - \frac{12.20 \times 3.80}{4.00 \times 12.00} = +\frac{41}{1{,}200}$$

The driven shaft will revolve, therefore, $1{,}200 \times 41/1{,}200 = 41$ r.p.m. in the same direction as the driving shaft.

$$W_1 = \frac{63{,}025 \text{ hp.}}{a \times \text{r.p.m.}} = \frac{63{,}025 \times 2}{4.10 \times 1{,}200} = 25.62 \text{ lb.}$$

$$W_2 = W_1\left(\frac{F - 2a}{F - S}\right) = 25.62\left(\frac{12.20 - 8.20}{12.20 - 12.00}\right) = 512.40 \text{ lb.}$$

$$W_3 = W_2 - W_1 = 512.40 - 25.62 = 486.78 \text{ lb.}$$

The pitch-line velocity of the sun gear S about its own axis is equal to $0.5236 \times 6.00 \times 41 = 129$ ft. per minute.

The pitch-line velocity of tooth engagement of the planet pinion Pf is equal to $0.5236 \times 6.10 \times 1{,}200 = 3{,}833$ ft. per minute.

The corresponding pitch-line velocity of the planet pinion Ps is equal to $3{,}833\dfrac{Ps}{Pf} = 3{,}641$ ft. per minute.

With a tooth load W_2 of 512.40 lb., the potential power transmitted by the sun gear S is equal to $512.40 \times 3{,}641 = 1{,}865{,}648$ ft.-lb. per minute. A power loss of 1 per cent due to tooth friction would amount to 18,656 ft.-lb. per minute.

With a tooth load W_3 of 486.78 lb., the potential power transmitted through the sun gear F and its planet pinion Pf is equal to $486.78 \times 3{,}833 = 1{,}865{,}828$ ft.-lb. per minute + 18,656 ft.-lb. per minute. The total potential power transmitted here is,

therefore, equal to 1,884,484 ft.-lb. per minute. A power loss of 1 per cent here would amount to 18,845 ft.-lb. per minute, and the total power loss for this drive would equal 37,501 ft.-lb. per minute, or about 1.14 hp. This would require an input of 3.14 hp. to deliver 2 hp. to the driven shaft.

One of the most compact arrangements of a planetary gear train employs an internal gear for one sun gear and a spur gear for the other, with but one planetary pinion that meshes with both sun gears. Either sun gear may be fixed. Generally, however, the internal sun gear is the fixed gear. In Fig. 102 is shown one construction of this type of planetary gear train with the internal sun gear as the fixed one. In this example,

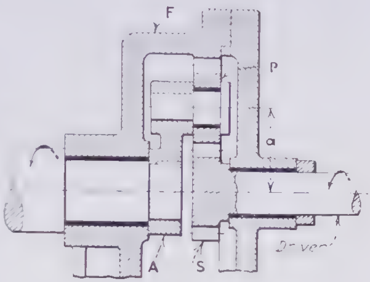


FIG. 102.—Simple form of planetary gear train.

F = fixed sun gear
S = moving sun gear
P = planet pinion meshing with both sun gears
A = arm which carries the planet pinion

We will first determine the reduction ratio of such a train. The analysis is made in the same manner as before. In the construction shown, the sun gear S is the driving member while the arm A is the driven member. As it is always simplest to determine the speed ratio between the two shafts when the arm A is the driving member, we will first consider this arm as driving. With all of the gears locked together and revolved for one turn, the driven shaft will make one revolution in the same direction as the driving member. Then, with the arm A locked, and the fixed sun gear F revolved backward for one turn to restore it to its original position, the sun gear S will move in the opposite direction, or farther ahead, in the direction of rotation of the driving arm A, an amount that depends upon the reduction or ratio between S and F considered as a simple train. This amount is equal to F/S, whence, the speed ratio when the arm A is driving is equal to $1 + \frac{F}{S} = \frac{S + F}{S}$. When the sun gear S is the driving member, the reduction ratio will be the reciprocal of this, or equal

to $\dfrac{S}{S+F}$. With this construction, the driving and driven shafts always turn in the same direction.

As a definite example, we will determine the reduction ratio of a planetary gear train, as shown in Fig. 102, with the following values:

$$F = 12.000 \text{ in.}$$
$$S = 2.000 \text{ in.}$$
$$P = 5.000 \text{ in.}$$
$$a = 3.500 \text{ in.}$$

$$\text{Reduction ratio} = \frac{S}{S+F} = \frac{2}{2+12} = \frac{1}{7}$$

We will now consider the tooth loads on this type of planetary gear train. In Fig. 103, a diagram of such a train is shown.

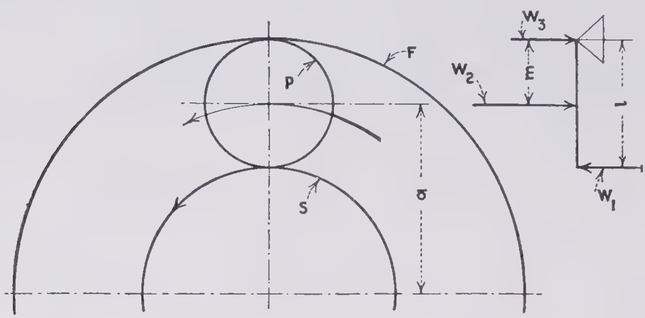


FIG. 103.—Diagrammatic form of gear train shown in Fig. 102.

As before, the action between these gears can be compared to a simple lever. In effect, the pitch point on the fixed sun gear F is the fulcrum, while the load is applied at the pitch point of the moving sun gear S to force the center of the planet pinion P ahead. At the right, in Fig. 103, is shown a diagram that represents these conditions. The force W_1 is the driving force, W_2 is the tangential load at the center of the planet pinion P which acts against the carrying arm A to force it ahead, while W_3 is the pressure against the fulcrum at the pitch point of the fixed sun gear F. As forces W_1 and W_2 are acting in opposite directions, the pressure W_3 is equal to their difference. We have, from the diagram in Fig. 103,

$$W_1 \times l = W_2 \times m$$
$$W_3 = W_2 - W_1$$

As l is the diameter of the planet pinion and m is its radius, $l = 2m$, whence,

$$W_2 = \frac{l}{m} W_1 = 2W_1$$

$$W_3 = W_2 - W_1 = W_1$$

When the power and speed requirements are known, these loads may be readily determined. Thus, when

S = pitch diameter of the driving gear
V = velocity in feet per minute of point of application of driving force

$$V = \frac{S \times \text{r.p.m.}}{12} = 0.2618S \times \text{r.p.m.}$$

$$W_1 = \frac{33{,}000 \text{ hp.}}{V} = \frac{126{,}050 \text{ hp.}}{S \times \text{r.p.m.}}$$

As a definite example, we will assume that the foregoing example of this type of planetary gear train is to transmit 2 hp. when the driving shaft revolves 1,200 r.p.m.

$$W_1 = \frac{126{,}050 \times 2}{2 \times 1{,}200} = 105.04 \text{ lb.}$$

$$W_2 = 2W_1 = 210.08 \text{ lb.}$$

$$W_3 = W_1 = 105.04 \text{ lb.}$$

The reduction ratio is $1/7$, whence the speed of the driven shaft will be $1/7$ of 1,200 or 171 r.p.m.

The pitch-line velocity of engagement on the fixed sun gear F will be equal to $0.2618 \times F \times 171 = 537$ ft. per minute.

The pitch-line velocity of engagement on the sun gear S will be the same as that on the fixed sun gear F, as the same pinion meshes with both, whence, the same number of teeth, or length of pitch line, must come in contact with both mating gears in the same interval of time.

With a tooth load of 105.04 lb., the potential power transmitted by the gear teeth is equal to $105.04 \times 537 = 56{,}406$ ft.-lb. per minute. This is slightly less than the actual power transmitted (66,000 ft.-lb. per minute), whence, the power loss here would be slightly less than on a simple train.

This construction of a planetary gear train, using a spur gear and an internal gear for the sun gears, is the only one that will compare favorably, as regards tooth-friction power losses, with

simple trains. The reduction ratios that can be obtained on this type of planetary train are very much smaller than those that can be obtained on other types; about 10:1 is the greatest ratio that it is practical to use, and even this ratio requires the use of a planet pinion several times as large as the spur sun gear. This type of planetary train may also be constructed with a double planet pinion, the one meshing with the spur sun gear while the other meshes with the internal sun gear. In Fig. 104, such a construction is shown.

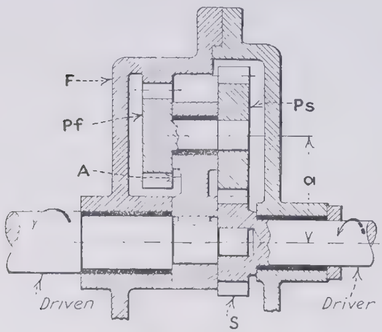


FIG. 104.—Double-planet type of gear train.

We will first determine the reduction ratio of such a train. This is accomplished in the same manner as before. In the construction shown, the sun gear S is the driving member, while the arm A is the driven member. As it is always simplest to determine the speed ratio between the two shafts when A is the driving member, we will consider this arm as driving and proceed as before.

With all of the gears locked together and revolved for one turn, the driven shaft will make one revolution in the same direction as the driving shaft. Then, with the arm A locked and the fixed sun gear F revolved backward for one turn to restore to its original position, the sun gear S will soon move in the opposite direction, or farther ahead in the direction of rotation of the driving shaft, an amount that depends upon the ratio of the gear train between them considered as a simple train. This amount is equal to $\frac{F \times Ps}{Pf \times S}$. Whence, the speed ratio when the arm A is driving is equal to

$$1 + \frac{F \times Ps}{Pf \times S} = \frac{Pf \times S + F \times Ps}{Pf \times S}.$$

When the sun gear S is the driving member, the reduction ratio will be the reciprocal of this, or equal to $\frac{Pf \times S}{(Pf \times S) + (F \times Ps)}$.

With this construction, the driving and driven members always turn in the same direction.

As a definite example, we will determine the reduction ratio of the planetary gear train shown in Fig. 104, with the following values:

$$\begin{aligned} F &= 10.000 \text{ in.} \\ S &= 2.000 \text{ in.} \\ Pf &= 3.000 \text{ in.} \\ Ps &= 5.000 \text{ in.} \\ A &= 3.500 \text{ in.} \end{aligned}$$

$$\text{Reduction ratio} = \frac{Pf \times S}{(Pf \times S) + (F \times Ps)} = \frac{3 \times 2}{(3 \times 2) + (10 \times 5)} = \frac{3}{28}$$

We will now consider the tooth loads on such a planetary gear train. In Fig. 105, a diagram of such a train is shown.

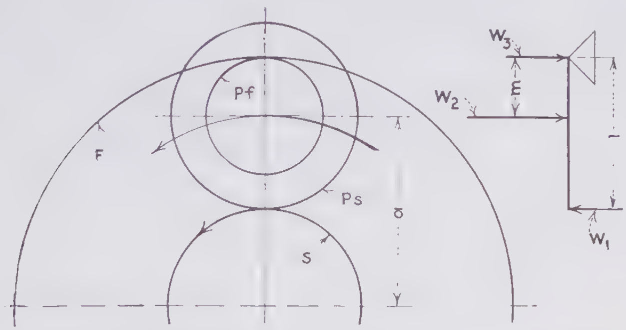


FIG. 105.—Diagrammatic form of gear train shown in Fig. 104.

As before, the action between these gears can be compared to a simple lever. In effect, the pitch point on the fixed sun gear F is the fulcrum, while the load is applied at the pitch point of the sun gear S to force the center of the planet pinions ahead. At the right, in Fig. 105, is shown a diagram that represents these conditions. The force W_1 is the driving force, W_2 is the tangential load at the center of the planet pinions, while W_3 is the pressure against the fulcrum at the pitch point of the fixed sun gear F. As forces W_1 and W_2 are acting in opposite directions, the pressure W_3 is equal to their difference.

$$\begin{aligned} W_1 \times l &= W_2 \times m \\ W_3 &= W_2 - W_1 \end{aligned}$$

From the diagram in Fig. 105, we have

$$m = \frac{Pf}{2} \text{ and } l = \frac{Ps + Pf}{2}$$

$$W_2 = \frac{b}{a} W_1 = W_1\left(\frac{Ps + Pf}{Pf}\right)$$

When the power and speed requirements are known, these loads may be readily determined. The value of W_1 is determined in the same manner as in the preceding example.

As a definite example, we will assume that the foregoing example of this type of planetary gear train is to transmit 2 hp. when the driving shaft revolves 1,200 r.p.m.

$$W_1 = \frac{126{,}050 \times 2}{2 \times 1{,}200} = 105.04 \text{ lb.}$$

$$W_2 = W_1\left(\frac{Ps + Pf}{Pf}\right) = 105.04\left(\frac{5 + 3}{3}\right) = 280.11 \text{ lb.}$$

$$W_3 = W_2 - W_1 = 280.11 - 105.04 = 175.07 \text{ lb.}$$

The reduction ratio is $3/28$, whence, the speed of the driven shaft will be $3/28$ of 1,200, or 129 r.p.m.

The pitch-line velocity of engagement on the fixed sun gear F will be equal to $0.2618F \times 129 = 338$ ft. per minute. The corresponding pitch-line velocity on the moving sun gear S will be equal to $338 \times \frac{Ps}{Pf} = 563$ ft. per minute.

With a tooth load of 105.04 lb., the potential power transmitted from the sun gear S to its mating pinion will be equal to $105.04 \times 563 = 59{,}138$ ft.-lb. per minute. This is slightly less than the actual power transmitted.

With a tooth load of 175.07 lb., the potential power transmitted from the planet pinion Pf to the fixed sun gear F will be equal to $175.07 \times 338 = 59{,}174$ ft.-lb. per minute, which is also less than the actual power. The tooth friction losses here will also be slightly less than for a simple train.

Differential gear trains are seldom used for the continuous transmission of any large amounts of power. Usually, they are employed either as a compensating device, such as in a rear-axle drive for an automobile, or to obtain small corrections of position, such as in some instruments and on some machine tools. Occasionally, however, they are employed for the continuous transmission of power, and in such cases, the tooth loads, velocity of

tooth engagements, and potential power transmitted should be carefully analyzed, because, generally, the potential power handled here is very much greater than the actual power transmitted.

The design of differential trains depends primarily upon the duty they have to perform. Sometimes they are in the form of a planetary gear train but with both sun gears movable. At other times, they appear as simple trains, and the large amount of potential power that must be handled is not always self-

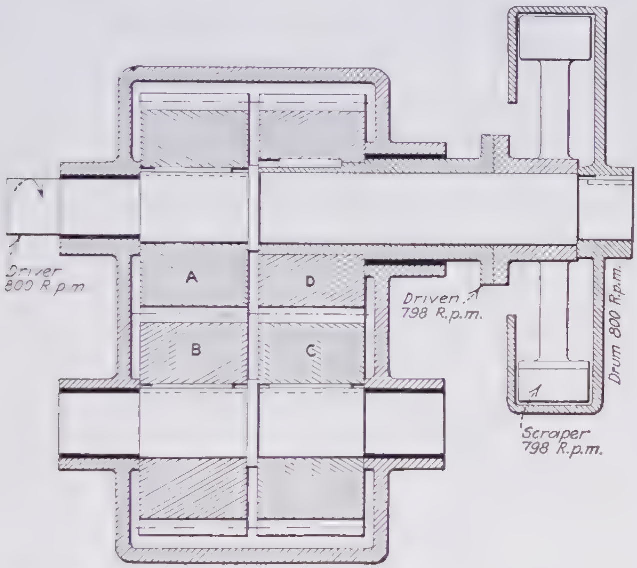


FIG. 106.—Differential gear train where potential power is large.

evident and is easily overlooked. Figure 106 shows an example of this kind.

This drive was required to impart a relatively slow movement to a spider carrying scrapers which operated inside of a revolving barrel or drum. The drum is about 16 in. in diameter and revolves about 800 r.p.m., while the scrapers revolve about 798 r.p.m., or 2 r.p.m. in relation to the drum. The actual power required to operate the scrapers is about 1 hp. The gears used in this train were all nearly the same size, which was about 10 in. in diameter. For example, with 27 teeth in A, 28 teeth in B, 30 teeth in C, and 29 teeth in D, the speed ratio between the

shaft driving the drum and the sleeve driving the scrapers would be $\frac{A \times C}{B \times D} = \frac{27 \times 30}{28 \times 29} = \frac{405}{406}$. Thus, when the gear A revolves 800 r.p.m., the gear D which is fastened to the sleeve that drives the scrapers would revolve very closely to 798 r.p.m.

The first construction of this gear train made no allowances for the high potential loads carried. In operation, the gears were completely destroyed in a very few minutes. The next set was made without a definite analysis of the conditions but was made heavy enough to carry about 75 hp. as a simple train. Superficially, this seemed ample, as the whole unit was driven with a 10-hp. motor, and the scrapers had but 1 hp. of actual work to perform. This set in operation showed rapid wear, while the heating of the gear case was so excessive that the unit could not be run under load for much more than 1 hour at a time. In commercial operation, this unit was to be operated nearly 24 hours per day. In addition, the friction losses were so great that full speed could not be obtained from the motor even when it was overloaded to the point where it developed nearly 13 hp.

The following analysis made apparent the cause of the difficulties. The mean diameter of the scraper blades is about 14 in., the speed of these scrapers in relation to the drum is but 2 r.p.m., giving them a working velocity of 7.33 ft. per minute. The load on these scrapers, when performing work equivalent to 1 hp., thus became about 4,500 lb. The pitch diameters of the gears were about 10 in., whence, the tooth loads on these gears would be approximately $^{14}/_{10} \times 4{,}500 = 6{,}300$ lb. The pitch-line velocity of tooth engagement on these gears, revolving together at about 800 r.p.m., is about 2,100 ft. per minute. Whence, the potential power transmitted by each pair amounts to $6{,}300 \times 2{,}100 = 13{,}230{,}000$ ft.-lb. per minute, or about 400 hp. The differential speed is about 1 revolution in 400, and the potential power increases over the actual power practically in direct proportion, a condition which is very similar to many planetary gear trains. The power loss here due to tooth friction, if 1 per cent of the potential power, amounts to 4 hp. on each pair, making a total of 8 hp. In this case, the input must be 9 hp., while the output is but 1 hp. of useful work.

This gear set was redesigned to handle 400 hp. in each pair, and the case was redesigned to dissipate 8 hp. in heat. The unit then ran successfully.

For years, inventors have been seeking a way to arrange a mechanical toothed gear train so that with a constant-speed driving member it would be possible to obtain any change of speed from zero up to any specified speed in either direction without shifting gears. Many ingenious combinations have been proposed, but one and all prove unsatisfactory upon close analysis. The desired range of speeds can be obtained, but they are obtained at the expense of a large loss of power in tooth friction and also give too low a torque at the low speeds to be useful.

This result seems to be inevitable by the very nature of the problem. In order to obtain such variations in speed with gears of fixed tooth numbers and ratios, a differential train must be used somewhere in the combination together with some form of friction drive. This differential, in effect, acts as the beam of a pair of weighing scales. The result is high potential powers and low actual powers. The output will be the difference between these partially balanced forces, minus a large tooth-friction loss.

All of such drives develop into uniform-torque drives. Usually, a gear reduction is employed to build up the torque at low speeds. A friction-controlled differential drive of this sort has its torque limited by the tooth load that the differential combination can carry. This load is practically constant, and the mechanical advantage between this differential and the final drive is also constant; hence, the maximum torque at the final drive is practically constant at all speeds. To make the parts of such a combination rugged enough to carry a large torque at low speeds would involve such a cumbersome unit that it would be impractical. Or, to put it another way, the same gears used in a simple gear train could transmit far more power than when arranged in combination with such a variable-speed, differential train.

The writer will not go so far as to say it cannot be done—that is too broad and sweeping a statement. To date, however, he has never seen any such combination that was efficient enough to justify its construction. If such a variable-speed control is necessary, there are other better and simpler ways of obtaining it more efficiently.

CHAPTER VIII

STRENGTH AND DURABILITY OF GEAR TEETH

The strength of gear teeth and the amount of power that can safely be transmitted by them are still open questions. So many variable and uncertain factors are involved that it is not surprising that a wide variety of different formulas and rules has been proposed from time to time. Few, if any, systematic experiments were made to obtain data on this subject until about 1911, when Prof. Guido H. Marx, of Stanford University, started a series of tests, the results of which have been published in several papers presented before the American Society of Mechanical Engineers.

The present object is to review existing data on the strength and durability of gear teeth. The purpose of gears is to transmit power efficiently, quietly, and with reasonable life. To accomplish this, three different factors must be considered: the strength of the tooth considered as a beam, the durability of the tooth surfaces in action under load, and the efficiency of the gears. These three factors will be considered individually.

BEAM STRENGTH OF GEAR TEETH

In 1879, John H. Cooper made an investigation of the subject of strength of gear teeth and found that there were then in existence about 48 well-established rules for horsepower and working strength, differing from each other, in extreme cases, about 500 per cent. Summing up, he selected the following formula for cast-iron gear teeth from an English rule published in 1868:

$$X = 2{,}000pf$$

Where X = breaking load on gear tooth, pounds

p = circular pitch of gear, inches
f = width of face of teeth, inches

In conclusion, he makes this pertinent remark:

It must be admitted that the shape of the tooth has something to do with its strength, and yet no allowance appears to have been made by the

rules tabulated above, the breaking strength being based upon the pitch or thickness of the teeth at the pitch line or circle, as if the thickness at the root of the teeth were the same in all cases as it is at the pitch line.

In 1886, Prof. William Harkness found from an examination of the bibliography of the subject, dating back to 1796, that, according to the constants and formulas used by various authors, there were differences of 15:1 in the power that could be transmitted by a given pair of gears. As a result of his investigations, he found that all of these formulas might be expressed in one of the three following forms:

$$\text{Horsepower} = CVpf \text{ or } CVp^2 \text{ or } CVp^2f$$

Where C = coefficient, or constant
V = pitch-line velocity, feet per second
p = circular pitch of gear, inches
f = width of face of teeth, inches

As a result of his examinations, Professor Harkness proposed the following formula for cast-iron gears:

$$\text{Horsepower} = \frac{0.910Vpf}{\sqrt{1 + 0.65V}}$$

The various factors that affect the strength of gear teeth and that should be included in a complete formula may be summed up as follows:

1. The physical characteristics of the material of which the gear is made.
2. The shape and size of the gear teeth.
3. The point on the gear-tooth profile where the maximum load is applied.
4. The consideration of whether the load is at any time carried by a single tooth or whether it is divided between two teeth.
5. The influence of velocity in causing the teeth to break by shock.
6. The influence of errors in the gear-tooth profiles in causing greater or less shock loads.
7. The influence of the rotating masses driven by the gear teeth in causing greater or less shock loads.
8. The character of the load transmitted, that is, whether the load is steady or suddenly applied or variable and subject to sudden overloads.
9. The factor of safety used to cover all the uncertainties of the other factors and to insure working stresses safely within the ultimate strength of the material employed.

Wilfred Lewis seems to have been the first to use the form of the gear tooth as one of the factors in a formula for the strength of

gear teeth. The Lewis formula which has become the one most widely used today was presented in a paper read before the Engineers' Club of Philadelphia on Oct. 15, 1892. This formula is as follows:

$$W = spfy \tag{79}$$

Where W = transmitted tooth load, pounds
s = safe working stress in material, pounds per square inch
p = circular pitch, inches
f = width of face of gear, inches
y = tooth-form factor

The factor y is obtained by considering the gear tooth as a beam, fixed at one end and loaded at the other. This factor may be obtained graphically, and in Fig. 107 this is done.

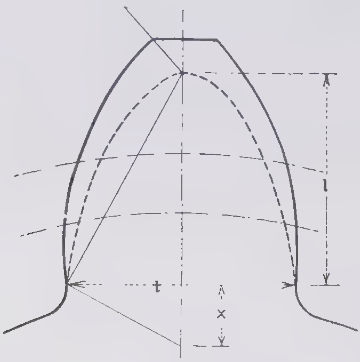


FIG. 107.—Graphical method of determining the Lewis form factor.

We have. for the strength of such a beam,

$$W = \frac{sft^2}{6l}$$

Where W = load, pounds
s = maximum fiber stress, pounds per square inches
f = width of beam, inches
t = thickness of beam, inches
l = length of beam, inches

It can be shown by similar triangles in Fig. 107 that

$$x = \frac{t^2}{4l}$$

Substituting this value in the equation, we have

$$W = sf\frac{2x}{3} \text{ or, for the gear tooth } W = spf\frac{2x}{3p}$$

whence,

$$y = \frac{2x}{3p}$$

These values for y for several different gear-tooth systems are assembled in Table XVII.

Wilfred Lewis makes the following statement in his paper:

> What fiber stress is allowable under different circumstances and conditions cannot be definitely settled at present, nor is it probable that any conclusion will be acceptable to engineers unless based upon carefully made experiments.

He then gives for use, in the absence of anything more definite, a table of safe working stresses for cast iron and steel for pitch-line velocities ranging from 100 to 2,400 ft. per minute. These values were established from a series of factors credited to E. R. Walker, Newcastle-under-Lyme, 1868. These factors were later put into the form of the following equation by Carl G. Barth:

$$s = \frac{600S}{600 + V} \tag{80}$$

Where s = safe working stress, pounds per square inch
S = safe static stress, pounds per square inch
= ⅓ elastic limit of the material
V = pitch-line velocity, feet per minute

Referring to the several factors that affect the strength of gear teeth, it will be seen that the Lewis formula includes the following:

1. Strength of material used in the gears.
2. Shape and size of gear teeth.
3. The point on the gear tooth where the load is applied is assumed to be at the tip. This is seldom true, but any error resulting from this assumption is on the safe side.
4. The load is assumed to be carried by one tooth. Any error resulting from this assumption is also on the safe side.
5. The influence of velocity.
9. A factor of safety of three is included in the value of the safe static stress.

The following factors have not been included.

6. Influence of the extent of errors in the gear-tooth profiles.
7. Influence of the rotating masses.
8. The character of the load.

As a matter of fact, for high-speed gears, the order of accuracy in the gear-tooth profiles is necessarily much higher than in commercial low-speed gear drives, in order to secure smooth operation, and the Barth equation, which determines the velocity factor, is often altered to the following:

$$s = \frac{1{,}200S}{1{,}200 + V} \tag{81}$$

In effect, this change in the constant from 600 to 1,200 introduces an accuracy factor.

In practice, the influence of the rotating masses and the influence of the character of the load are taken care of through the use of larger factors of safety, determined partially by experience. In general, the Lewis formula has proved satisfactory in practice. Generally, the safe loads as determined by this formula are smaller than the loads that actually can be transmitted safely, as has been proved by successful installations carrying theoretical excess loads.

In 1911, Prof. Guido H. Marx undertook a series of tests to obtain more definite data on this subject. Cast-iron gears were driven at various pitch-line velocities and loaded with a prony brake until the teeth failed. His first paper, covering tests up to pitch-line velocities of 600 ft. per minute, was presented before the A.S.M.E., in December, 1912. His second paper, written in collaboration with Prof. Lawrence E. Cutter, was presented before the same society in September, 1915. In this second paper were given the results of a continuation of these tests up to a pitch-line velocity of 2,000 ft. per minute.

A third paper, by Lloyd J. Franklin and Charles H. Smith, gave the results of further tests made under Professor Marx's direction with gears of varying degrees of accuracy consistent with commercial production of cast-iron gears.

The first two series of tests were planned to determine the following:

1. The values of velocity coefficients at pitch-line speeds from 0 to 2,000 ft. per minute on cast-iron, 14½-deg. composite-tooth form and 20-deg. stub-tooth form gears.
2. The values of arc-of-action coefficients for the same tooth form.
3. Experimental values of tooth-form factors (Lewis y factor).

TABLE XVII.—VALUES OF THE BEAM FACTOR y FOR USE IN THE LEWIS FORMULA

Number of teeth	Gear-tooth system			
	14½-deg. composite and generated	20-deg. full depth	20-deg. stub	14½-deg. variable center distance
10	0.056	0.064	0.083	0.131
11	0.061	0.072	0.092	0.128
12	0.067	0.078	0.099	0.125
13	0.071	0.083	0.103	0.123
14	0.075	0.088	0.108	0.121
15	0.078	0.092	0.111	0.120
16	0.081	0.094	0.115	0.120
17	0.084	0.096	0.117	0.120
18	0.086	0.098	0.120	0.120
19	0.088	0.100	0.123	0.119
20	0.090	0.102	0.125	0.119
21	0.092	0.104	0.127	0.119
23	0.094	0.106	0.130	0.119
25	0.097	0.108	0.133	0.118
27	0.099	0.111	0.136	0.116
30	0.101	0.114	0.139	0.114
34	0.104	0.118	0.142	0.112
38	0.106	0.122	0.145	0.110
43	0.108	0.126	0.147	0.108
50	0.110	0.130	0.151	0.110
60	0.113	0.134	0.154	0.113
75	0.115	0.138	0.158	0.115
100	0.117	0.142	0.161	0.117
150	0.119	0.146	0.165	0.119
300	0.122	0.150	0.170	0.122
Rack........	0.124	0.154	0.175	0.124

From the results of these tests, the following formulas were developed:

For $14\frac{1}{2}$-deg. composite-tooth gears,

$$W = \frac{1}{k}spf\left(0.154 - \frac{1.26}{n}\right)va$$

For 20-deg. stub-tooth gears,

$$W = \frac{1}{k}spf\left(0.278 - \frac{2.69}{n}\right)va$$

Where W = safe tooth load, pound
s = modulus of rupture = 36,000 lb. per square inch for cast iron
p = circular pitch, inches
f = width of face of gear, inch
n = number of teeth in gear
v = velocity coefficient
a = arc-of-action coefficient
k = factor of safety

The following factors of safety are suggested by Professor Marx:

k = 4, for steady loads with no reversal of stress
k = 6, for suddenly applied loads with no reversal of stress
k = 8, for suddenly applied loads with reversal of stress

Table XVIII gives values for the velocity coefficient v and the arc-of-action coefficient a, which were determined from these experiments.

The Marx and Cutter formula introduces a factor for duration of contact, different factors of safety for different kinds of loads, and experimental values for the tooth form and velocity factors; otherwise, it is essentially the same as the Lewis formula. These experiments showed that the tooth-form factors as calculated by Lewis were smaller than those determined by actual breaking tests; that a greater duration of contact increased the load required to break the teeth; and that the velocity factor, as given by the Barth equation, was appreciably different from that obtained in actual tests. The test values were smaller at low pitch-line velocities and greater at the higher velocities than those obtained by means of the Barth equation.

TABLE XVIII.—VALUES OF v AND a FOR USE IN THE MARX AND CUTTER FORMULA

Values of v						Values of a			
Pitch velocity, feet per minute	14½-deg. composite system	20-deg. stub-tooth system	Pitch velocity, feet per minute	14½-deg. composite system	20-deg. stub-tooth system	Teeth in engaging gears		14½-deg. composite system	20-deg. stub-tooth system
						Single-tooth engages			
0	1.000	1.000	1,000	0.485	0.550	..	..	1.00	1.00
100	0.795	0.825	1,100	0.470	0.540	12	12	1.10	1.13
200	0.730	0.755	1,200	0.455	0.525	20	30	1.15	1.20
300	0.675	0.705	1,300	0.445	0.515	30	30	1.47	1.22
400	0.635	0.665	1,400	0.435	0.505	30	40	1.60	1.24
500	0.595	0.635	1,500	0.430	0.495	30	60	1.60	1.25
600	0.565	0.615	1,600	0.420	0.485	30	80	1.60	1.26
700	0.540	0.595	1,700	0.415	0.475	30	100	1.60	1.27
800	0.520	0.580	1,800	0.410	0.470	30	Rack	1.60	1.29
900	0.500	0.565	1,900	0.405	0.460	100	100	1.60	1.31
1,000	0.485	0.550	2,000	0.400	0.450	100	Rack	1.60	1.33

In the discussion that followed the presentation of the first paper by Professor Marx, Ralph E. Flanders remarked:

It is also important to know how much the accuracy of the cutting affects the strength of the gears at high speeds. All grades of accuracy are used in commercial work.

In 1924, Franklin and Smith undertook a series of tests with 60-tooth, 10-d.p., cast-iron gears made to varying degrees of accuracy. The first series of gears had errors of the order of 0.001 in.; the second series, 0.002 in.; and the third series, 0.006 in. The tests were run on the same apparatus as that used by Professor

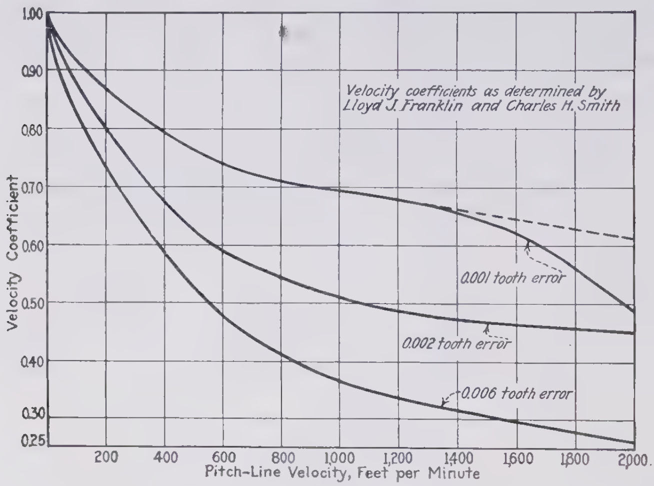


FIG. 108.—Variation of velocity coefficients with magnitude of gear errors

Marx, and the results were presented in a paper read before the A.S.M.E., in December, 1924.

These tests showed that the accuracy of the gears had a marked influence on their strength. Table XIX gives the velocity coefficients established from these tests, while in Fig. 108, diagrams of these same results are presented.

The form of the curve in Fig. 108, for the most accurate of the gears tested, shows a downward trend at the higher speeds quite different from the other two curves. The probable cause of this is the fact that the testing machine was strained beyond its capacity in making these tests. As noted in the log of the tests, the steel driving gears on the testing machine were almost entirely destroyed under the heavy loads imposed. With stronger and more rigid testing equipment, this curve would

TABLE XIX.—VALUES FOR VELOCITY COEFFICIENTS FOR GEARS HAVING DIFFERENT ERRORS (*Franklin and Smith*)

Error in gears, inches	Pitch-line velocity in feet per minute										
	0	100	200	300	400	500	600	700	800	900	1,000
0.001	1.000	0.917	0.863	0.822	0.794	0.766	0.741	0.724	0.711	0.703	0.695
0.002	1.000	0.875	0.788	0.725	0.675	0.625	0.588	0.562	0.544	0.526	0.512
0.006	1.000	0.825	0.732	0.652	0.587	0.525	0.476	0.440	0.412	0.387	0.364
	1,100	1,200	1,300	1,400	1,500	1,600	1,700	1,800	1,900	2,000	
0.001	0.687	0.680	0.672	0.654	0.641	0.620	0.592	0.560	0.525	0.488	
0.002	0.500	0.491	0.483	0.476	0.470	0.465	0.461	0.457	0.453	0.450	
0.006	0.352	0.340	0.328	0.317	0.306	0.296	0.287	0.278	0.270	0.262	

probably have a very similar form to the others, as indicated by the dotted line. This downward trend in the first curve also indicates the possible influence of the errors in the train of driving gears on all of the results obtained. Probably all of the gears would show higher values if the drive were smoother, such as would be obtained by means of a belt drive to a pulley fastened directly on the shaft of one of the test gears. These tests indicate that the strength of gear teeth in a train of gears is less than when the gears operate as a single pair.

The velocity factor has always been considered as constant for any given speed, regardless of the characteristics of the material in the gear or the amount of the unit load. In the absence of any evidence to the contrary, this assumption results in simple forms of equations. All tests made to determine this velocity factor have been made by running cast-iron gears to destruction, and the velocity factors so obtained have been used for all other materials.

The Increment Load.—The thought has been advanced that the actual tooth load in operation is the combination of two loads: first, the transmitted or useful load; and second, the additional or increment load caused by inaccurate tooth profiles, suddenly applied loads, and so on. If it were possible to produce theoretically perfect teeth, the impulse delivered to the driven gear would be smooth and continuous, fulfilling ideal conditions never attained in practice. With a pair of such gears made of inelastic material, variations in the driven velocity would be impossible, and the calculated static load could then be carried at any desired pitch-line speed, provided the gears were in perfect balance.

Tooth action, however, is made up of accelerations and decelerations due to errors in tooth form and spacing. At low speeds, these errors have a relatively slight effect, but at high speeds, they may develop increment loads many times greater than the applied or useful load. The gear teeth, therefore, must be sufficiently strong to carry this increment load in addition to the applied load.

In an article published in *Zeitschrift des Vereines Deutscher Ingenieure*, in 1899, Oscar Lasche, of Berlin, discussed the probable effects of errors and the large increment loads that may result from them at high pitch-line velocities. He gave certain calculated values for these increment loads but stated:

All such figures, however, depend upon assumptions which influence the results to a large extent and do not permit the determination of results accurately.

Considering rigid materials, he stated that the increment load caused by errors would be proportional to the square of the pitch-line velocities. He went on to say, however:

The more elastic the teeth are, the greater the errors that can be permitted. The differences in the velocities caused by the errors can be partially absorbed by the teeth themselves, without disturbing the velocities of the rotating masses so much, and without causing such high increment loads. The duration of the changes in velocity is spread over a longer period of time because of the springy action of the teeth, and consequently the acceleration of the masses is reduced and the increment load is cut down.

Recent Tests on Lewis Gear-testing Machine.—In a paper read before the British Institute of Mechanical Engineers, in May, 1916, Daniel Adamson discussed the probable value of the increment loads along similar lines to those followed by Lasche. As a result of correspondence between Daniel Adamson, Wilfred Lewis, and Charles H. Logue, Mr. Lewis proposed, in a paper read before the A.S.M.E., in December, 1923, the construction of a testing machine that would enable these increment loads to be measured. A special research committee was organized by the Society, and arrangements were made to build the testing machine and to have a series of tests conducted at the Massachusetts Institute of Technology. The first series of tests with this machine were started, in 1925, by John E. Nicholas, and the results are described in his thesis "The Influence of Errors and Elasticity on the Strength of Gear Teeth," submitted in 1926.

The machine[1] consists of a pair of test gears and pinions mounted in a swing frame centered in the axis of the pinion shaft and supported at a convenient distance upon weighing scales. By this means the torque can be measured when transmitting power received from a belt on a flywheel pulley mounted on the pinion shaft. The test gears are mounted on a telescopic sleeve and shaft, in which an initial torque can be introduced through a connecting nut of long pitch at the outer end of the telescope. The elastic reaction of the telescope maintains any desired aver-

[1] For complete description, see article by Wilfred Lewis, *American Machinist*, vol. 59, p. 875.

age load on the gear teeth, which is augmented and reduced in action by the inaccuracies in the teeth and by the speed. Insulated ball bearings are used throughout, and electric circuits are established through telephone receivers which may be interrupted when either test gear fails to make contact with its pinion. When starting, and at slow speeds, there will be no interruption, but as the speed is increased, there will come a time when the inaccuracies cause a momentary break in tooth contact. This will be announced by one of the telephone receivers, and at that moment the corresponding speed, as indicated by the tachometer, is noted. Another initial load will show a different breaking speed, and in this way, the loads and speeds can be correlated by experiment. For rigid teeth in rolling contact, the initial load required to maintain contact is proportional to the speed squared, and the departure from this relation may be ascribed to elasticity and mass effects. For the accurate determination of the latter, special provision has been made in the application of a torsion balance to the pinion shaft direct. A light rod of tempered steel is attached to the pinion shaft and anchored at its outer end. A capillary pen mounted in a holder traversed by the oscillation of the pinion shaft moves over a paper ribbon traveling at a known rate of speed. This device was designed by H. H. Williams. It makes possible the accurate determination of the oscillating period for the pinion shaft alone, with or without one or two of the flywheels, and with or without each of the test gears.

Provision has also been made to multiply and record the inaccuracies in the teeth on circular diagrams in which they will appear as radial displacements, and a novel feature of this mechanism is that diagrams can be made and compared for the same teeth under very heavy loads as well as under the ordinary comparatively light loads which cannot show the effects of elastic deflection and compression.

The original purpose of these tests was to determine the effect of errors on the strength of gear teeth, but the results of the first tests showed that the elastic properties of the materials had such a pronounced influence that the whole subject of the effect of errors was too large to be completely investigated in a single series of tests. In fact, in order to interpret the results obtained on these tests, it has been found necessary to make a study of several phases of the "dynamics of elastic bodies," an interesting

and important subject that seems to have escaped the attention it deserves.

A series of four of these studies has been issued as *Progress Reports* of the A.S.M.E. Special Research Committee on Gears. These consist of studies, first, of the influence of elasticity on perfect gears; second, of the influence of acceleration loads on imperfect gears; third, elastic impact; fourth, mass effect of rotating masses. The analysis of elastic impact was the work of Carl G. Barth, while all of the others were worked out on lines suggested by Mr. Barth. Tentative equations were derived which seem to be reasonably consistent with the test data. The following abstracts from these reports give the essential features:

Perfect Gears.—Involute gears made of rigid materials, rigidly mounted, and with perfect profiles perfectly spaced would transmit power smoothly and continuously at any speed without any variation in the total pressure between the teeth, regardless of whether a single pair or two pairs of mating teeth were in contact.

With elastic materials, however, there would be a certain amount of deformation caused by the load. This deformation would be due partly to the bending of the teeth and partly to the compression of the material. This distortion or deformation, furthermore, would not be constant at all phases of the tooth mesh.

This variation in the amount of deformation is due to several causes. First, the point of contact travels over the active profiles of the mating teeth, thus applying the load at different distances from the base of the teeth and causing different amounts of bending. Second, at some positions, two pairs of mating teeth would be sharing the load, while at other positions but a single pair of mating teeth would be in action, so that the whole load would be concentrated on them.

The foregoing conditions would result in a variation in the amount of deformation as the contact traveled over the active profiles of the teeth. Even with perfectly formed and spaced gear teeth, therefore, the elasticity of the material would cause a variation in the smoothness of the flow of power.

When these gears are operated under load, this variation in deformation would tend to accelerate and decelerate the gears and, because of their inertia, would result in an increasing and decreasing tooth load as well as a corresponding variation in the speed of the gears.

The acceleration would take place as the contact left the portion of the profile where the deformation was greatest to engage that part of the profile where the deformation was less. This condition would exist when the contact shifted from a single pair of mating teeth to two pairs of mating teeth or at the overlap of the tooth action. This acceleration would tend to increase the load on the portion of the profile where the static deformation was less and act to smooth out the action by increasing the deformation at this part of the profile. Thus, at low speeds, where the influence of inertia is less, the variations in velocity would follow very closely the variations in the static deformation, but as the speeds increased and the effects of inertia became greater, the variations in velocity would become less and less, until a balanced condition was reached, where the effects of inertia were sufficient to cause a uniform deformation of the tooth profile at all points of contact. At such a speed, the gears would travel at a constant velocity, although a variation in the tooth load would still be present. This variation in the tooth load would be the difference in the static load required to distort all points of the tooth profile equally under the specific conditions of mesh.

It should be pointed out that the acceleration load would be imposed principally, if not entirely, upon the undeflected tooth profile as it came into action, so that, with perfect gears, the acceleration load due to deformation probably would not increase the load on a single tooth to an amount greater than the transmitted load. The total load would be increased, but this increased load would be distributed over two pairs of teeth. On the other hand, the reaction from the deceleration load when one pair of teeth is leaving contact would probably be very nearly equal to the acceleration load, and this reaction, or the force required to restore the lost velocity, would be carried by a single pair of teeth in addition to the transmitted load.

If the acceleration is sufficient to cause the teeth to leave contact, they will come together again with an impact which would impose a heavier additional load than either the acceleration or the reaction from the deceleration. With perfect gears and a constant applied load, however, such an impact is not possible, because the additional loads caused by the deformation will always be less than the applied load and, hence, will not be sufficient to overcome the influence of this applied load. With

imperfect gears, however, we must deal with both accelerations and impact loads.

Influence of Errors on Acceleration Loads.—Errors on gear-tooth profiles, assuming a constant velocity of the driving member, act to accelerate and decelerate the driven member. This varying velocity of the driven member results in a varying load on the gear teeth, the amount of this variation depending largely upon the masses of the revolving parts, the nature and extent of the errors, and the velocities of the gears. If the gears were made of rigid materials, the acceleration load would vary as the square of the velocity. With elastic materials, however, as the acceleration load increases, the deformation of the tooth profiles will also increase, and this deformation will increase most where the acceleration load is greatest and, hence, tend to reduce this load by reducing the acceleration.

It appears, from a study of the diagrams or charts which measure the accuracy of the gears on the testing machine, that the effective error seems to act while the load is being transferred from one pair of mating teeth to the next pair. The errors may be on the tooth profiles or in the spacing, but their influence seems always to be most during the transfer of the load from tooth to tooth.

When a positive error is present, the engaging pair of teeth will tend to take over the entire load. When the error is greater than the deformation under load, one pair of teeth only will carry the load, except during the time that the load is being transferred from tooth to tooth.

In the foregoing, the assumption was first made that the velocity of the driving member is constant and that all variation in velocity takes place on the driven member. Actually, the variation in velocity will be divided between the two members, depending upon their relative effective masses. Thus, if these masses are equal, the variation in velocity will be divided equally between them, and the resultant mass influence on the acceleration load will be equal to one-half of either equal mass.

Separation of Teeth because of Acceleration.—At the instant that the second pair of mating tooth profiles are carrying the load and the acceleration force has ceased to act, the masses are moving apart from each other because of the difference in velocity imparted to them by the acceleration force which slows down the driving member and speeds up the driven member. This relative

movement apart is resisted by the applied load. If the materials were rigid, the maximum amount of this separation would be equal to the product of the acceleration force and the distance through which it acts, or the amount of the effective error, divided by the applied load. Thus, when the acceleration force is equal to the applied load, the amount of separation would be equal to the effective error. The influence of elasticity, however, on this separation is to reduce its amount.

The analysis is based upon the assumption that but a single pair of teeth is carrying the load. When we consider the actual conditions, however, we find that we have a much more complex condition to deal with, because at one instant two pairs of mating teeth may be sharing the load, while at another instant but a single pair will be in action. Although, for relatively large errors, a single pair of teeth will carry practically all of the load, yet when the error is small or when the deformation becomes great enough, two pairs of teeth will be sharing these loads, and the exact distribution of these loads will always be uncertain.

The analysis is also based on the assumption that uniform accelerations result from deformations and errors. This assumption is probably seldom exactly true, yet the influence of the elasticity of the materials and the inertia of the revolving parts will tend to make the acceleration approach this condition. Furthermore, as the additional work done during acceleration is used to determine the amount of separation, any error in the foregoing assumption will have but a very small influence on the accuracy of the final results. The results of actual tests seem to indicate that the amount of this error is the primary factor, while its exact nature seems to have but little influence on the results.

Impact Loads.—When an error is present which is greater than the deformation of the material, one pair of teeth only will carry the load, except as the contact shifts from one pair of teeth to the next. When the acceleration caused by such an error is sufficient to cause the teeth to leave contact with each other, they will come together again with an impact the force of which may be many times greater than the applied or transmitted load. If the load on these same gears were increased so that the acceleration did not cause the teeth to leave contact with each other but caused a zero load for an instant, the maximum load on the teeth would then be a suddenly applied load whose intensity would be double the applied load. If the applied load were increased still

further on these same gears, the acceleration would not be sufficient to reduce the minimum instantaneous load to zero, and we would have a suddenly applied load of less intensity, which would be greater than the applied load but less than double this load. The critical load in all cases is the maximum one, and this maximum load will be the impact load.

Influence of Rotating Masses.—Before the foregoing can be applied to the actual determination of the loads on gear teeth, it is necessary to determine the influence of the rotating masses. The effective mass at the pitch line of any gears attached to shafts carrying other rotating masses is variable. This variation will depend largely upon the speed and accuracy of the gears and the length and diameter of the connecting shafts. This variation in the effective mass is caused by the elasticity of the connecting member. If the shaft were rigid, and all the masses were rigidly connected to it, the effective mass would be constant, and all variations in velocity resulting from errors in the gear-tooth profiles would be imparted to all of the connected rotating bodies. The inertia of these bodies would thus cause a higher tooth load than that caused by the masses of the gear blanks alone, and the greater the mass of these connected bodies the greater this additional tooth load would be.

The shafts or other connections are not rigid, however, but are elastic, so that when it takes less load to twist the connection than to accelerate the connected masses, the shaft or other connection will twist and the acceleration of the connected masses will be correspondingly reduced.

A complete analysis of the load conditions on any pair of gears thus requires the following determinations:

1. Effective mass at pitch line of gears.
2. Acceleration load.
3. Amount of separation.
4. Impact load.

The equations developed for this purpose as a result of the several studies on the dynamics of elastic bodies are as follows. These equations probably give a fair measure of the truth, as they explain and check quite consistently the test data obtained on the Lewis gear-testing machine. As many approximations and assumptions were necessary to complete the analysis, it is probable that the results of further tests which are to be made will result in some further refinement.

Determination of Effective Mass.—When

m = effective mass at pitch line of gears
m_1 = effective mass at pitch line of pinion
m_2 = effective mass at pitch line of gear

$$m = \frac{m_1 \times m_2}{m_1 + m_2} \tag{82}$$

When either member of the pair is connected to a shaft carrying other rotating masses, the effective mass of that member at its pitch line should be determined for the particular load and speed involved. We will first assume that the rotating masses are attached to the gear shaft. The calculations would be identical in either case. Thus, when

m_a = effective mass attached to gear shaft at radius equal to pitch line of gear
m_b = mass effect of m_a at pitch line of gear
m_g = effective mass of gear blank itself at pitch line
m_p = effective mass of pinion blank at pitch line
V = pitch-line velocity, feet per minute
Z = elasticity factor of connecting member
e = error in tooth action, inches
f = applied load, pounds per inch of face
d_t = static deformation of tooth profiles under applied load
p = circular pitch, inches
R_1 = pitch radius of pinion
R_2 = pitch radius of gear

Then,

$$m_2 = m_g + m_b \tag{83}$$

where

$$m_b = \frac{\sqrt{B^2 + 4AC} - B}{2A} \tag{84}$$

$$A = 0.0025p\left(\frac{R_1 + R_2}{R_1 \times R_2}\right)^2 m_a V^2 \tag{85}$$

$$B = (m_p + m_g)A + Zm_p\left(e - \frac{d_t}{2}\right) \tag{86}$$

$$C = Zm_a m_p\left(e - \frac{d_t}{2}\right) \tag{87}$$

$$d_t = f\left(\frac{E_1 z_1 + E_2 z_2}{E_1 z_1 \times E_2 z_2}\right) \tag{88}$$

Where z_1 = elasticity form factor of pinion tooth
z_2 = elasticity form factor of gear tooth
E_1 = modulus of elasticity of pinion material
E_2 = modulus of elasticity of gear material

Values of the elasticity form factors of gear teeth are given in Table XX.

TABLE XX.—ELASTICITY FORM FACTORS FOR GEAR TEETH

Number of teeth	14½-deg. involute tooth		20-deg. full-depth tooth		20-deg. stub tooth	
	y	z	y	z	y	z
12	0.067	0.09206	0.078	0.09659	0.099	0.10315
13	0.071	0.09382	0.083	0.09837	0.103	0.10417
14	0.075	0.09545	0.088	0.10000	0.108	0.10537
15	0.078	0.09659	0.092	0.10121	0.111	0.10604
16	0.081	0.09768	0.094	0.10179	0.115	0.10690
17	0.084	0.09871	0.096	0.10235	0.117	0.10731
18	0.086	0.09936	0.098	0.10289	0.120	0.10791
19	0.088	0.10000	0.100	0.10341	0.123	0.10849
20	0.090	0.10061	0.102	0.10392	0.125	0.10886
21	0.092	0.10121	0.104	0.10442	0.127	0.10922
23	0.094	0.10179	0.106	0.10490	0.130	0.10975
25	0.097	0.10262	0.108	0.10537	0.133	0.11026
27	0.099	0.10315	0.111	0.10604	0.136	0.11075
30	0.101	0.10367	0.114	0.10669	0.139	0.11122
34	0.104	0.10442	0.118	0.10752	0.142	0.11168
38	0.106	0.10490	0.122	0.10830	0.145	0.11212
43	0.108	0.10537	0.126	0.10904	0.147	0.11241
50	0.110	0.10582	0.130	0.10975	0.151	0.11291
60	0.113	0.10648	0.134	0.11042	0.154	0.11336
75	0.115	0.10690	0.138	0.11107	0.158	0.11387
100	0.117	0.10731	0.142	0.11168	0.161	0.11425
150	0.119	0.10772	0.146	0.11226	0.165	0.11472
300	0.122	0.10830	0.150	0.11282	0.170	0.11529
Rack.....	0.124	0.10868	0.154	0.11336	0.175	0.11584

When the connecting shaft is cylindrical and of uniform diameter, its elasticity factor can be calculated. If it is of varying size, or if a flexible coupling is used, this factor may be determined experimentally. Thus, when

P = load applied to twist connecting member at a radius equal to the pitch radius of gear, pounds

T = torsional deflection at radius equal to pitch radius of gear, inches

$$Z = \frac{P}{T} \tag{89}$$

The torsional deflection of a solid cylindrical shaft is given by the following equation:

$$T = \frac{P}{E_t} \times \frac{32R^2L}{\pi d^4} \tag{90}$$

Where T = torsional deflection at radius R
P = load at radius R, pounds
R = radius where load is applied, inches
L = length of shaft, inches
d = diameter of shaft, inches
E_t = torsional modulus of elasticity

Thus, for a solid cylindrical shaft, we have

$$Z = \frac{\pi d^4 E_t}{32R^2L} \tag{91}$$

As a definite example, we will assume that we have a 3-d.p. cast-iron pinion, 30 teeth, 14½-deg. involute tooth form, with a 3-in. face driving a 90-tooth cast-iron gear mounted on a 6-in. shaft with a 10-in. length between it and a connected mass. We will also assume a total tooth load of 1,500 lb., or 500 lb. per inch of face, and a pitch-line velocity of 1,000 ft. per minute. We will also assume an error in tooth action equal to 0.003 in. and the masses as follows:

$$m_a = 20.00$$
$$m_p = 1.00$$
$$m_g = 3.00$$

We will first solve Eq. (91), for which we have the following values:

d = 6.000
E_t = 12,600,000 (torsional modulus of elasticity of steel)
R = 15.000
L = 10.000

whence,

$$Z = \frac{3.1416 \times 1{,}296 \times 12{,}600{,}000}{32 \times 225 \times 10} = 712{,}512$$

We will next solve Eq. (88), for which we have

$$\begin{aligned} f &= 500 \\ E_1, E_2 &= 15{,}000{,}000 \\ z_1 &= 0.10367 \\ z_2 &= 0.10715 \end{aligned}$$

whence,

$$d_e = 500 \times 0.000001265 = 0.00063250$$

We will now solve Eq. (84), for which we have

$$\begin{aligned} p &= 1.0472 \\ R_1 &= 5.000 \\ R_2 &= 15.000 \\ V &= 1{,}000 \\ e &= 0.003 \end{aligned}$$

$$\begin{aligned} A &= 0.003723V^2 = 3{,}723 & &\text{(see Eq. (85))} \\ B &= 4A + 976 = 14{,}953 & &\text{(see Eq. (86))} \\ C &= 19{,}520 & &\text{(see Eq. (87))} \end{aligned}$$

whence,

$$m_b = 0.997$$

Then, from Eq. (83), we have

$$m_2 = m_g + m_b = 3.000 + 0.997 = 3.997$$

And from Eq. (82),

$$m = \frac{1.000 \times 3.997}{1.000 + 3.997} = 0.800$$

In order to show the influence of velocity on the effective mass Table XXI has been computed from the foregoing data, except for changes in velocity of from 0 to 5,000 ft. per minute pitch-line velocity. An inspection of the values of m_b in this table shows that the influence of the rotating mass mounted on the gear shaft decreases quite rapidly as the speed increases and has practically disappeared entirely at the higher speeds.

When the rotating mass is attached to the shaft of the driving member or pinion, a condition which exists when the pinion is mounted on the motor shaft or coupled directly to it, the effective

mass of the pinion must be determined in the same manner as before. The equations then become as follows:

$$m_1 = m_p + m_b \tag{92}$$

$$m_b = \frac{\sqrt{B^2 + 4AC} - B}{2A} \tag{93}$$

$$A = 0.0025p\left(\frac{R_1 + R_2}{R_1 \times R_2}\right)^2 m_a V^2 \tag{94}$$

$$B = (m_p + m_g)A + Zm_g\left(e - \frac{d_t}{2}\right) \tag{95}$$

$$C = Zm_a m_g\left(e - \frac{d_t}{2}\right) \tag{96}$$

TABLE XXI.—INFLUENCE OF VELOCITY ON EFFECTIVE MASS

V	m_b	m_2	m
0	20.000	23.000	0.958
100	12.351	15.351	0.939
200	7.325	10.325	0.912
300	4.922	7.922	0.889
400	3.560	6.560	0.868
500	2.704	5.704	0.851
600	2.125	5.125	0.837
700	1.713	4.713	0.825
800	1.408	4.408	0.815
900	1.176	4.176	0.807
1,000	0.997	3.997	0.800
1,200	0.740	3.740	0.789
1,400	0.569	3.569	0.781
1,600	0.450	3.450	0.775
1,800	0.364	3.364	0.771
2,000	0.300	3.300	0.767
2,500	0.198	3.198	0.762
3,000	0.140	3.140	0.758
3,500	0.104	3.104	0.756
4,000	0.080	3.080	0.755
4,500	0.063	3.063	0.754
5,000	0.051	3.051	0.753

As a definite example, we will assume the same values as before, except that the diameter of the pinion shaft is 3.000 in. and the effective mass of the connected rotating part at the pitch line of the pinion is equal to 10.00. This gives the following values:

$$
\begin{aligned}
m_a &= 10.000 \\
m_p &= 1.000 \\
m_g &= 3.000 \\
d &= 3.000 \\
E_t &= 12{,}600{,}000 \\
R &= 5.000 \\
L &= 10.000 \\
d_t &= 0.00063250
\end{aligned}
$$

We will first solve Eq. (91), whence,

$$
Z = \frac{3.1416 \times 81 \times 12{,}600{,}000}{32 \times 25 \times 10} = 400{,}790
$$

The solution of Eq. (88) is the same as before. We will now solve Eq. (93), for which we have

$$
\begin{aligned}
A &= 0.00186169V^2 = 1{,}862 \\
B &= 4A + 2{,}847 = 10{,}295 \\
C &= 28{,}466
\end{aligned}
$$

whence,

$$
m_b = 2.024
$$

Then, from Eq. (92), we have

$$
m_1 = 1.000 + 2.024 = 3.024
$$

and from Eq. (82),

$$
m' = \frac{3.000 \times 3.024}{3.000 + 3.024} = 1.506
$$

Table XXII has been calculated to show the influence of the velocity on the effective mass, as before. Here, also, the influence of the rotating mass attached to the pinion shaft decreases rapidly with increasing speed and practically disappears at the higher speeds.

In actual practice, rotating masses are usually attached to both the gear and pinion shafts. In this case, we would determine the effective mass at the pitch line of each gear as before and then determine the mass influence at the pitch line by the use of Eq. (82). As a definite example, we will assume the same values as before, with the connected masses as specified in the two pre-

ceding examples. Then, at a pitch-line velocity of 1,000 ft. per minute, we should have

$$m_1 = 3.024$$
$$m_2 = 3.997$$

whence,

$$m = \frac{3.024 \times 3.997}{3.024 + 3.997} = 1.722$$

TABLE XXII.—INFLUENCE OF VELOCITY ON EFFECTIVE MASS

V	m_b	m_1	m
0	10.000	11.000	2.357
100	9.132	10.132	2.314
200	7.669	8.669	2.229
300	6.232	7.232	2.120
400	5.116	6.116	2.038
500	4.257	5.257	1.910
600	3.589	4.589	1.814
700	3.064	4.064	1.726
800	2.645	3.645	1.645
900	2.304	3.304	1.572
1,000	2.024	3.024	1.506
1,200	1.593	2.593	1.391
1,400	1.286	2.286	1.297
1,600	1.056	2.056	1.220
1,800	0.881	1.881	1.156
2,000	0.745	1.745	1.103
2,500	0.514	1.514	1.006
3,000	0.374	1.374	0.942
3,500	0.283	1.283	0.899
4,000	0.221	1.221	0.868
4,500	0.181	1.181	0.847
5,000	0.145	1.145	0.829

Table XXIII is given to show the variation in the mass influence under these last conditions as the speed varies. In this case, the mass influence at very low speeds is about ten times the mass influence of the gear blanks alone, while, at the higher speeds, it is only about 15 per cent greater than for the gear blanks alone.

On important gear drives, where the actual mass conditions can be determined, the mass influence should be calculated directly, as shown. In many cases, however, these mass condi-

tions are not always definitely known. In such cases, only the masses of the gear blanks can be determined. The foregoing example represents an average mass condition, and from it mass factors have been determined for use when only the masses of the gear blanks are known. In use, the value of the effective mass of the gears at their pitch line, as established from Eq. (82), would be multiplied by the factor corresponding to the pitch-line velocity to be used. These mass factors are given in Table XXIII.

TABLE XXIII.—APPROXIMATE MASS FACTORS

V	m_1	m_2	m	Mass factor
0	11.000	23.000	7.441	10.000
100	10.132	15.351	6.104	8.203
200	8.669	10.325	4.712	6.332
300	7.232	7.922	3.781	5.081
400	6.116	6.560	3.165	4.253
500	5.257	5.704	2.736	3.677
600	4.589	5.125	2.421	3.254
700	4.064	4.713	2.182	2.932
800	3.645	4.408	1.995	2.681
900	3.304	4.176	1.845	2.480
1,000	3.024	3.997	1.722	2.314
1,200	2.593	3.740	1.531	2.058
1,400	2.286	3.569	1.393	1.872
1,600	2.056	3.450	1.288	1.731
1,800	1.881	3.364	1.206	1.621
2,000	1.745	3.300	1.141	1.533
2,500	1.514	3.198	1.028	1.382
3,000	1.374	3.140	0.956	1.285
3,500	1.283	3.104	0.908	1.220
4,000	1.221	3.080	0.874	1.175
4,500	1.181	3.063	0.852	1.145
5,000	1.145	3.051	0.833	1.120

Effective Mass of Gear Blanks.—If the gear blank is a solid disk, its effective weight at the pitch line is equal to about one-half its actual weight. Its effective mass would be equal to its effective weight divided by the force of gravity, or 32.2.

The effective weight of larger gears with webs or spokes will be greater, however, as more of the weight is concentrated in the rim. In extreme cases, the effective weight at the pitch line

will be very nearly equal to the actual weight of the gears themselves. The effective weight at the pitch line may be determined in any specific case by determining the radius of gyration. Then, when

$$\begin{aligned} k &= \text{radius of gyration} \\ W &= \text{weight of gear blank} \\ R &= \text{pitch radius of gear} \\ w &= \text{effective weight at pitch line} \end{aligned}$$

$$w = \frac{R^2}{k^2}W \tag{97}$$

In many cases, it is desirable to have a general formula covering the average conditions rather than specific ones which require exact weights, radii of gyration, etc., for their solution. Such a formula is given by Charles H. Logue in the "*American Machinists' Gear Book*" which covers the weight of unfinished gear blanks. The weight of the finished gear is found by deducting 30 per cent. This would give the following approximation for the weight of finished gears:

$$W = p^2nT \tag{98}$$

where

$$\begin{aligned} W &= \text{weight, pounds} \\ p &= \text{circular pitch, inches} \\ n &= \text{number of teeth} \\ T &= \text{width of face, inches} \end{aligned}$$

This formula has its limitations and cannot be used for low numbers of teeth or for very large gears where there is a large variation in design of blanks.

The effective weight of the gear at its pitch line, as noted before, will also have a different relation to the total weight with a change in design. For smaller gears of plain disk form, this effective weight will be equal to about one-half of the total weight, while for very large gears with spokes and heavy rims, it may be very nearly equal to the total weight. The following equation would represent a fair approximation to such a change in the disposition of the masses:

$$\text{Effective weight factor} = 0.50\left(\frac{0.002R^2 + 1}{0.001R^2 + 1}\right) \tag{99}$$

where R = pitch radius of gear

Then, by combining Eqs. (98) and (99), we should have

$$w = 0.50p^2nT\left(\frac{0.002R^2 + 1}{0.001R^2 + 1}\right) \tag{100}$$

where $w =$ effective weight of gear at its pitch line
whence

$$m_p = 0.0155p^2nT\left(\frac{0.002R_1^2 + 1}{0.001R_1^2 + 1}\right) \tag{101}$$

and

$$m_g = 0.0155p^2nT\left(\frac{0.002R_2^2 + 1}{0.001R_2^2 + 1}\right) \tag{102}$$

where m_p = effective mass of pinion at pitch line
m_g = effective mass of gear at pitch line

Determination of Acceleration Loads.—When

f_a = force of acceleration, pounds
f = applied load, pounds per inch of face
p = circular pitch, inches
R_1 = pitch radius of pinion, inches
R_2 = pitch radius of gear, inches
m = effective mass at pitch line of gears
V = pitch-line velocity, feet per minute
e = error in tooth action, inches
d_t = deformation of tooth profiles under applied load
T = width of face of gears, inches

Then

$$f_a = \frac{f_1 \times f_2}{f_1 + f_2} \tag{103}$$

where

$$f_1 = 0.0025p\left(\frac{R_1 + R_2}{R_1 \times R_2}\right)^2 mV^2 \tag{104}$$

$$f_2 = fT\left(\frac{e}{d_t} + 1\right) \tag{105}$$

$$d_t = f\left(\frac{E_1z_1 + E_2z_2}{E_1z_1 \times E_2z_2}\right) \qquad \text{(see Eq. (88))}$$

where z_1, z_2 = elasticity form factor of gear (see Table XX)

E_1, E_2 = modulus of elasticity of material

As a definite example, we will use the same pair of gears as before, whence, we have

$$
\begin{aligned}
f &= 500 \\
p &= 1.0472 \\
R_1 &= 5.000 \\
R_2 &= 15.000 \\
m &= 1.722 \\
V &= 1{,}000 \\
e &= 0.0030 \\
T &= 3.000 \\
z_1 &= 0.10367 \\
z_2 &= 0.10715 \\
E_1, \text{ and } E_2 &= 15{,}000{,}000
\end{aligned}
$$

From Eq. (88), we have

$$d_t = 0.00063250$$

From Eq. (104), we have

$$f_1 = 321$$

From Eq. (105), we have

$$f_2 = 8{,}615$$

From Eq. (103), we have

$$f_a = 309$$

Determination of Amount of Separation.—When k = amount of separation in inches, and all other factors are as before,

$$k = \frac{f_a}{fT}e - \left[\left(\frac{f_a}{fT}\right)^2 + 1\right]\frac{d_t}{2} \tag{106}$$

As a definite example, we will use the same data as before, whence,

$$
\begin{aligned}
f &= 500 \\
T &= 3.000 \\
f_a &= 309 \\
e &= 0.0030 \\
d_t &= 0.00063250
\end{aligned}
$$

whence, from Eq. (106), we have

$$k = 0.000288$$

The value of k may be either plus or minus. When the value is plus, it indicates an actual separation of the mating tooth

profiles. When the value is minus, it indicates that the teeth do not actually leave contact with each other but that some deformation exists at the instant when the direction of the load is reversed after the influence of the acceleration has ceased to act. In the foregoing example, this value of k is plus.

Determination of Impact Load.—When F = total maximum load of impact, and all other factors are as before,

$$F = fT\left(1 + \sqrt{1 + \frac{2k}{d_t}}\right) \tag{107}$$

As a definite example, we will use the same data as before, whence,

$$\begin{aligned} f &= 500 \\ T &= 3{,}000 \\ d_t &= 0.00063250 \\ k &= 0.000288 \end{aligned}$$

whence, from Eq. (107),

$$F = 3{,}573$$

Thus, on these gears, when running at a pitch-line velocity of 1,000 ft. per minute with a total transmitted load of 1,500 lb., the maximum load is about 3,600 lb., or over twice the applied load.

The foregoing gives the results to date of the tests on single pairs of gears on the Lewis gear-testing machine. According to this analysis, the additional or increment load on gear teeth is not directly proportional to the applied load but is almost a constant for any given mass and speed condition when the applied load is appreciable. If further tests substantiate these preliminary results, it would be possible to compute tables of increment loads for various conditions, which would make the determination of the equivalent static load for any operating load a very simple process. In the meantime, the foregoing data may prove helpful in checking critical gear drives, particularly when the present methods of calculating the safe tooth loads do not seem to check with the actual conditions.

Load Conditions on Trains of Gears.—Thus far, we have attempted to analyze the load conditions on a single pair of gears only. On the testing machine, it has been apparent that the back, or master, gears were subjected to heavier loads than the test pair. This same condition is also often apparent on

the second pair of a double-reduction gear set and likewise on the second pair of a train of gears. The conditions which may exist on such trains are so involved that anything like a complete analysis would be extremely complex. For the present, therefore, only a general survey will be attempted.

Where a gear train with two or more tooth meshes is involved, and with the power applied smoothly to the first mesh, it is apparent that, due to elasticity and errors in the tooth form and spacing, the power will be applied to the second mesh in impulses of greater or less intensity. The timing of these impulses may be such that, at times, the resultant load on the teeth at the second mesh may be more or may be less than if they were operating as a single pair. The probability is that, at some time during the cycle, the maximum impulse from the first mesh will be added to the acceleration impulse of the second mesh or to the impact load of the second mesh. In either case, the result would be a higher tooth load at the second mesh than would be present in a single pair.

All of the acceleration or impact impulse of the first mesh, however, would not be transmitted to the second mesh, because of the elasticity of the gear blank and its mounting and also because of the work absorbed in internal friction at the first mesh, etc. If a double-reduction gear train is involved, the elasticity of the shaft connecting the driven gear of the first mesh with the driving gear of the second mesh would act to reduce the intensity of the impulse transmitted from the first mesh even more. This condition might be analyzed in a very similar manner to that used to analyze the influence of the rotating masses mounted on the gear shafts.

The effective mass of the intermediate gear of a train is another uncertain factor. Any tendency of this gear to retard because of conditions at the second mesh is resisted by the driving impulses set up at the first mesh, thus probably making the effective mass of the intermediate gear at the second mesh greater than it would be otherwise. In some ways, it would seem as though this increased effective mass of the intermediate gear would influence both the first and second meshes practically alike, but experience seems to indicate that its primary influence is evident only at the second mesh. The cumulative effect seems to be carried along in the direction of the flow of power. Actual tests are now under consideration to obtain data on this

subject. At present, the best practice is to use a larger factor of safety for the gears beyond the first mesh of a train.

SUMMARY OF PRESENT PRACTICE

Present practice is based upon the assumption that the increment load is directly proportional to the applied load. For beam strength, the Lewis formula, as follows, is generally used:

$$W = spfy \qquad \text{(See Eq. (79))}$$

where W = transmitted tooth load, pounds
s = safe working stress of material, pounds per square inch
f = width of face of gears, inch
p = circular pitch, inch
y = tooth-form factor (see Table I)

The safe working stress of the material is usually determined by the use of the Barth equation, modified to suit the accuracy of the gears. This Barth equation is as follows:

$$s = \left(\frac{A}{A + V}\right)S \tag{108}$$

where s = safe working stress of material, pounds per square inch
S = safe static strength of material, pounds per square inch
A = factor, depending upon accuracy of gears

The following factors are commonly used in this equation:

A = 600 for ordinary commercial gears
A = 1,200 for carefully cut gears

The following modification of the Barth equation has been adopted by the American Gear Manufacturers' Association for use on high-speed gears running at pitch-line velocities of 4,000 ft. per minute and over. The order of accuracy on such high-speed gears must be very high, else the noise of operation would be objectionable.

$$s = \left(\frac{78}{78 + \sqrt{V}}\right)S \tag{109}$$

A summary of the foregoing velocity factors is given in Table XXIV.

The tensile strength of the material is divided by a factor of safety to determine the value of S, or the safe static strength of the material. The following factors of safety are suggested for use with these equations:

TABLE XXIV.—VELOCITY FACTORS

V	$\frac{600}{600 + V}$	V	$\frac{1,200}{1,200 + V}$	V	$\frac{78}{78 + \sqrt{V}}$
100	0.857	1,000	0.545	4,000	0.553
200	0.750	1,200	0.500	4,200	0.545
300	0.667	1,400	0.461	4,400	0.540
400	0.600	1,600	0.429	4,600	0.535
500	0.545	1,800	0.400	4,800	0.530
600	0.500	2,000	0.375	5,000	0.525
700	0.461	2,200	0.353	5,200	0.520
800	0.429	2,400	0.333	5,400	0.515
900	0.400	2,600	0.316	5,600	0.510
1,000	0.375	2,800	0.300	5,800	0.506
1,100	0.353	3,000	0.286	6,000	0.502
1,200	0.333	3,200	0.273	6,200	0.498
1,300	0.316	3,400	0.261	6,400	0.494
1,400	0.300	3,600	0.250	6,600	0.490
1,500	0.286	3,800	0.240	6,800	0.486
1,600	0.273	4,000	0.231	7,000	0.482
1,700	0.261			7,200	0.479
1,800	0.250			7,400	0.475
1,900	0.240			7,600	0.472
2,000	0.231			7,800	0.468
				8,000	0.465
				8,200	0.462
				8,400	0.459
				8,600	0.456
				8,800	0.454
				9,000	0.451
				9,200	0.448
				9,400	0.446
				9,600	0.443
				9,800	0.441
				10,000	0.438

For steady loads on single pairs of gears.................. 3
For suddenly applied loads on single pairs................ 4
For steady loads on gears of a train beyond the first mesh. 5
For suddenly applied loads on gears of a train beyond the first mesh.. 6

The following list gives values of the tensile strength for various materials which may be used:

Material	Tensile Strength, Pounds per Square Inch
Cast iron	24,000
Semisteel	36,000
Bronze	36,000
Cast steel (S.A.E. 1,235)	45,000
Forged steel (S.A.E. 1,030)	60,000
Forged steel (S.A.E. 1,045)	90,000
Forged steel (S.A.E. 3,245)	120,000

When the physical properties of the material are definitely known, the specific tensile strength of the particular material would naturally be used.

As further definite examples, we will use two gear drives which have proved satisfactory in service. The first will be called Example A, and the second, Example B. We will calculate the load conditions on these drives both by present practice as given in the foregoing and by use of the equations developed from the results of the tests on the Lewis gear-testing machine.

Example A.—A steady load of 175 hp. is transmitted at a pitch-line velocity of 380 ft. per minute. The pinion is of S.A.E. 1,045 steel, hardened, while the gear has a forged-steel rim of the same material. These are commercial gears of 2 d.p., 14½-deg. composite tooth form, of 20 and 60 teeth. The extent of the maximum error in action is about 0.006 in. The width of face is 6 in.; whence, we have the following values for Eq. (79) and 108):

$$W = \frac{175 \times 33{,}000}{380} = 15{,}197 \text{ (transmitted load)}$$

$$S = \frac{90{,}000}{3} = 30{,}000$$

$$\begin{aligned} p &= 1.5708 \\ f &= 6.000 \\ y &= 0.090 \text{ (for pinion)} \\ A &= 600 \\ V &= 380 \end{aligned}$$

whence, from Eq. (108), we have

$$s = \left(\frac{600}{600 + 380}\right)S = 18{,}360$$

and from Eq. (79),

$$W = 18{,}360 \times 1.5708 \times 6 \times 0.090$$
$$= 15{,}573 \quad \text{(calculated safe load)}$$

In this example, the calculated safe load is slightly greater than the actual transmitted load. The velocity factor in this example $\left(\frac{600}{600 + V}\right)$ is equal to 0.612, whence, the equivalent static load or the maximum working load due to impact, etc., would be equal to 15,197/0.612, which is equal to 24,832 lb.

We will now calculate the maximum impact load by using Eqs. (82) to (88), and (101) to (107). For these equations, we have the following values:

$$f = \frac{15{,}197}{6} = 2{,}533$$

$$\begin{aligned}
p &= 1.5708 \\
R_1 &= 5.000 \\
R_2 &= 15.000 \\
V &= 380 \\
n &= 20 \\
N &= 60 \\
e &= 0.006 \\
T &= 6.000 \\
z_1 &= 0.10061 \\
z_2 &= 0.0648 \\
E_1, E_2 &= 30{,}000{,}000
\end{aligned}$$

From Eqs. 101 and 102, we have

$$\begin{aligned}
m_p &= 4.70 \\
m_g &= 16.29
\end{aligned}$$

From Eq. 82, multiplied by the mass factor from Table XXIII for the specified pitch-line velocity, we have

$$m = 4.420\left(\frac{m_p \times m_g}{m_p + m_g}\right) = 16.124$$

From Eq. 104, we have

$$f_1 = 645$$

From Eq. 88, we have

$$d_t = 0.001621$$

From Eq. 105, we have

$$f_2 = 71{,}441$$

From Eq. 103, we have

$$f_a = 639$$

From Eq. 106, we have

$$k = -0.000560$$

whence, from Eq. (107), we have

$$F = 23{,}662$$

This last value of F, or the impact load, corresponds to the previous calculated value of 24,842, or the equivalent static load on the gear teeth. In this example, these two values are practically identical.

Example B. A steady load of 300 hp. is transmitted at a pitch-line velocity of 2,200 ft. per minute. The pinion is of S.A.E. 1,045 steel, while the gear is made with a forged-steel rim of the same material. These gears are 3 d.p., 20-deg. full-depth, involute tooth form with 21 and 84 teeth. The extent of the maximum error in action is about 0.002 in. The width of face is 14 in. This gives the following values for Eqs. (79) and (108):

$$W = \frac{300 \times 33{,}000}{2{,}200} = 4{,}500 \text{ (transmitted load)}$$

$$S = \frac{90{,}000}{3} = 30{,}000$$

$$p = 1.0472$$
$$f = 14.000$$
$$y = 0.104 \quad \text{(for pinion)}$$
$$A = 1{,}200$$
$$V = 2{,}200$$

whence, from Eq. (108), we have

$$s = \left(\frac{1{,}200}{1{,}200 + V}\right)S = 10{,}590$$

and from Eq. (79), we have

$$W = 10{,}590 \times 1.0472 \times 14 \times 0.104 = 16{,}147$$

In this example, the calculated safe load, as far as the beam strength of the tooth is involved, is about three and one-half

times as great as the transmitted load. Later, we will check these same gears for their safe load for wear which is often much less than it is for beam strength. The velocity factor in this example $\left(\frac{1{,}200}{1{,}200 + V}\right)$ is equal to 0.353, whence, the equivalent static load or maximum working load due to impact, etc., would be equal to 4,500/0.353, which is equal to 12,748 lb.

We will now calculate the maximum impact load, using Eqs. (82), (88), and (101) to (107). For these equations, we have the following values:

$$f = \frac{4{,}500}{14} = 321$$

$$\begin{aligned} p &= 1.0472 \\ R_1 &= 3.500 \\ R_2 &= 14.000 \\ V &= 2{,}200 \\ n &= 21 \\ N &= 84 \\ e &= 0.002 \\ T &= 14.000 \\ z_1 &= 0.10442 \\ z_2 &= 0.11138 \\ E_1 \text{ and } E_2 &= 30{,}000{,}000 \end{aligned}$$

From Eqs. (101) and (102), we have

$$\begin{aligned} m_p &= 5.06 \\ m_g &= 23.27 \end{aligned}$$

From Eq. (82), multiplied by the mass factor from Table XXIII for the specified pitch-line velocity, we have

$$m = 1.473\left(\frac{m_p \times m_g}{m_p + m_g}\right) = 4.156$$

From Eq. (104), we have

$$f_1 = 169$$

From Eq. (88), we have

$$d_t = 0.000200$$

From Eq. (105), we have

$$f_2 = 49{,}500$$

From Eq. (103), we have

$$f_a = 168$$

From Eq. (106), we have

$$k = -0.000025$$

whence, from Eq. (107), we have

$$F = 8{,}397$$

This last value of F corresponds to the previous value of 12,748 or the equivalent static load on the gear teeth. In this example, the second calculated value is only about two-thirds of the first computed value. The difference between these two values is no greater, however, than the present uncertainty as to the maximum load on gear teeth in action.

Non-metallic Pinions.—In order to reduce noise and vibration, particularly of high-speed gears, the pinion is often made of some non-metallic material, such as rawhide, fiber, bakelite, micarta, or fabroil. Such pinions are extensively used for electric motor pinions to drive other metal gears.

The characteristics of these materials are so different from the characteristics of metals, that gears employing these non-metallic materials should be considered by themselves. Due to their low modulus of elasticity, most of the effects of errors in tooth form and spacing are absorbed at the surface of the gear teeth and have relatively little effect on the strength of the gears. In addition, the profiles of the teeth of these non-metallic gears tend to conform to the conjugate form of their mating teeth very quickly, thus reducing still further any effects of errors in tooth form. In these installations, it is of advantage to have the number of teeth in the metal gear an even multiple of the number of teeth in the non-metallic pinion, so that the form of the teeth in this pinion will conform to the smallest possible number of mating teeth.

Very little definite experimental data are available as to the strength of such gears in operation. Tests so far made on the Lewis machine at the Massachusetts Institute of Technology with such gears have given negative results; that is, the effects of very small errors on the hardened-steel master gears have apparently been greater than the effects of about ten times larger errors on the non-metallic gears. Under these conditions, the effects of inaccuracies become of lesser importance.

The following velocity factor, which seems to check very well with results in practice, has been adopted by the American Gear Manufacturers' Association for these materials:

$$\text{Velocity factor} = \frac{150}{200 + V} + 0.25$$

whence,

$$s = \left(\frac{150}{200 + V} + 0.25\right) S \qquad (110)$$

The safe static strength S of bakelite and micarta is usually taken as 6,000 lb. per square inch. Values for the safe working stress of these materials for pitch-line velocities up to 3,000 ft. per minute are tabulated in Table XXV. Other than using these special-stress values, the calculations for the strength of these non-metallic gear teeth will be the same as those for metal gears.

Table XXV.—Safe Working Stress for Non-metallic Pinions

V	s	V	s
100	4,500	1,200	2,143
150	4,071	1,300	2,100
200	3,750	1,400	2,063
250	3,500	1,500	2,029
300	3,300	1,600	2,000
350	3,136	1,700	1,974
400	3,000	1,800	1,950
450	2,885	1,900	1,929
500	2,786	2,000	1,909
600	2,625	2,200	1,875
700	2,500	2,400	1,846
800	2,400	2,600	1,821
900	2,318	2,800	1,800
1,000	2,250	3,000	1,781
1,100	2,192		

WEAR ON GEAR TEETH

In addition to the problem of the beam strength of gear teeth, that of wear must be solved before the proper proportions of any gears to meet definite requirements of load and speed can be established. Until quite recently, this subject of wear has received but little attention. In the past few years, however,

several series of experiments have been made to obtain data on this subject. Much work still remains to be done before we can hope to have a very definite knowledge of the subject.

It has long been realized that, on high-speed gears in particular, wear is a much more critical factor than the beam strength of the teeth. This conclusion has led to the use of equations based on a limiting load per inch of face. An equation of this kind, commonly used in the United States, is the following:

$$\frac{W}{f} = KD \tag{111}$$

where $\frac{W}{f}$ = load per inch of face, pounds

K = constant depending upon load conditions

D = diameter of pinion, inches

A maximum value for W/f is also usually established, depending upon the diametral pitch of the gears, the material, and the allowable bearing loads on standard gear cases. The following values of K are often employed for heat-treated steel gears:

K = 62.5 for single-reduction gears, constant service

K = 100.0 for single-reduction gears, constant service, when the full load is reached only occasionally

Another similar equation that is widely used in England is the following:

$$\frac{W}{f} = K\sqrt{D} \tag{112}$$

In this case, for heat-treated steel gears, the following values of K are often used:

K = 175 for single-reduction gears, constant service

K = 250 for single-reduction gears, constant service, when the full load is reached only occasionally

No allowance is made in these equations for the effect of pitch-line velocity. They are based on experience with high-speed gears in service. When the gears are made accurately enough to operate quietly at about 5,000-ft.-per-minute pitch-line velocity, there seems to be but little difference in either their quietness or load-carrying ability between that speed and up to 8,000 or 10,000 ft. per minute.

Joseph Chilten, in a paper entitled "Toothed Gearing," Feb. 11, 1919, Newcastle-on-Tyne, gave a formula and constants

used by an Italian manufacturer, Luigi Pomini, Castellanza, Milan. These are as follows:

$$\frac{W}{f} = Kp \frac{1{,}480}{v + 32.8}$$

where $\frac{W}{f}$ = load per inch of tooth face for cast-iron gears, pounds

K = factor found in Table XXVI depending upon number of teeth in pinion and the reduction ratio

p = circular pitch, inches

v = pitch-line velocity, feet per second

These factors were deduced from observations on the wear of lubricated teeth with circular pitches varying from ½ to 2½ in.

About 1920, E. R. Ross, of the Warner Gear Company, started a series of wear tests on hardened-steel gears used in automobile transmissions. A report of these tests was presented before the American Gear Manufacturers' Association, in October, 1921. He stated in this report:

> A 50-hp. load was applied to the transmission by the motor dynamometer continuously for 6 hours. This represents an overload for this particular set of gears, being quite beyond anything to which they would be subjected in actual service. Preliminary runs demonstrated, however, that some such overload was necessary if we were going to have any measurable amount of wear in a test of reasonable length of time.

He then presented the data of these tests but drew no general conclusions.

A somewhat similar series of tests were made by Prof. C. W. Ham and J. W. Huckert of the University of Illinois, starting in 1922. The results of these tests were published in *Bull.* 149 of the Engineering Experiment Station of that university, in July, 1925. These tests covered both the efficiency and the wear of spur gears. The test pinions were made of steel, and the test gears of cast iron. Here, again, in order to accelerate the rate of wear, the tooth loads used were far beyond those that would be used in actual practice. Summing up the results of the tests on durability, the authors stated:

> The results of these tests indicate that the factors that have the greatest influence on the durability of gear teeth are lubrication, sliding action, vibration, and load transmitted. The number of tests in which the speed was varied and the range of speed used were not sufficient to justify the drawing of any conclusions as to the effect of speed.

In all cases where failure occurred, the cast-iron gear teeth broke down apparently by crushing the material in the region of the pitch line. In no case was there any indication of failure near the outer ends of these teeth other than a slightly pitted condition of the surfaces.

The unhardened-steel pinion teeth, on the other hand, failed by abrasive wear, which, in general, resulted in a final outline of double curvature.

Surface pressure is the most important of the factors that affect durability. Apparently for any pair of gears there is a critical surface pressure, governed by the properties of the materials, above which the life of the gears is short and below which the gears will run indefinitely without wear.

In December, 1925, a paper by Professors Marx, Cutter, and Green, of Stanford University, on "Some Comparative Wear Experiments on Cast-iron Gear Teeth" was presented before the A. S. M. E. In these tests, also, heavy overleads were used to obtain accelerated results. The results of these tests followed very closely those of the other similar tests.

TABLE XXVI.—VALUES OF K FOR USE WITH POMINI EQUATION

Number of teeth in pinion	Reduction ratio							
	1:1	1:2	1:3	1:4	1:5	1:6	1:8	1:10
12	2.80	3.40	3.80	4.20	4.36	4.54	4.80	5.00
14	3.20	3.80	4.20	4.60	4.88	5.08	5.40	5.60
16	3.50	4.20	4.64	5.06	5.36	5.58	5.84	6.10
18	3.80	4.40	5.00	5.40	5.76	5.96	6.24	6.44
20	4.20	4.90	5.40	5.90	6.20	6.40	6.88	6.90
24	5.00	5.76	6.30	6.80	7.04	7.30	7.60	7.80
28	5.70	6.40	7.04	7.60	7.88	8.14	8.50	8.64
32	6.40	7.28	7.92	8.40	8.80	9.04	9.40	
36	7.20	8.10	8.76	9.24	9.60	9.88		
40	7.90	8.84	9.56	10.28	10.44			

A large amount of experimental work still remains to be done before we can hope to have a very definite knowledge of the wear on gear teeth. First, it is of importance to establish definitely whether or not there is, for any pair of gears, a critical load, beyond which wear is very rapid and inside which wear is relatively slight, as intimated by Professor Ham. Second, if this critical load exists, it is of great importance to determine its value for

any pair. Third, if and when this critical load has been established, it would be of great importance to determine the rate of wear inside of such a critical load. In all previous tests, the rate of wear beyond this critical point has been measured, and the shape into which the tooth profile wore was considered primarily. This last feature is of secondary importance in many ways. The Lewis gear-testing machine, now at the Massachusetts Institute of Technology, lends itself very readily to tests of this nature, and the Special Research Committee on Gears of the A.S.M.E. hopes to have such tests conducted as soon as the opportunity offers itself.

Charles H. Logue, in the "American Machinists' Gear Book," published in 1910, suggested the use of the radius of curvature of the gear-tooth profile as a measure of the stresses on gear teeth that affect the wear. Joseph Jandesek followed this up in articles published in 1920 to 1922, giving numerous formulas, diagrams, and calculations based on the Hertz equation and using the maximum surface pressure, or compressive stress, as a measure of the wearing qualities.

About 1920, the writer first used the Hertz equation as a measure of gear-tooth wear, and in May, 1926, presented before the American Gear Manufacturers' Association a paper in which were given constants for various combinations of materials that had proved generally satisfactory during about 7 years' use. An abstract of this paper follows.

The contact conditions between spur-gear-tooth profiles are similar to the contact between two cylinders, except that on gear-tooth profiles the radius of curvature is constantly changing. If we use the contact and pressure conditions between two cylinders, therefore, as a measure of the stresses on the surfaces of gear teeth, we must first select some definite part of the gear-tooth profile as a basis of comparison.

In many cases, wear on gear teeth first becomes apparent at or near the pitch line. Possibly, one contributing cause to this effect is the fact that one tooth is usually carrying the entire load when contact is made on this part of the profile, while when contact takes place near the top or bottom of the active profile, two teeth are usually sharing the load. In the absence of definite evidence to the contrary, this supposition gives us reasonable cause to select the radius of curvature of the tooth profile at the pitch line as the one to use as a basis of comparison with the Hertz equation.

When two cylinders with parallel axes are brought into contact with each other, contact takes place along a line that is the tangent element of the two cylinders. When pressure is applied, this line broadens out into a surface as the cylinders are distorted. The greater the pressure the wider this surface will be. The pressure between the surfaces thus created is not uniformly distributed: the stress is greatest where the distortion is greatest, that is, along the line where contact was first made. These stresses reduce to zero along the line where contact ceases. This condition is illustrated in Fig. 109. The distances along the

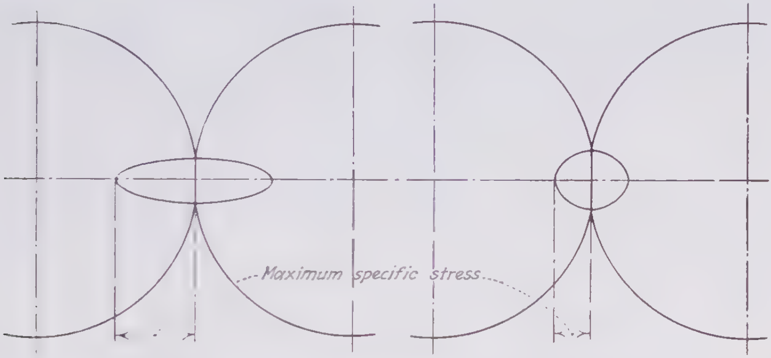


FIG. 109. FIG. 110.

FIG. 109.—Stress distribution with two hard-metal cylinders in contact.
FIG. 110.—Stress distribution with two soft-metal cylinders in contact.

horizontal axis represent the intensity of the pressures between the surfaces. The area of the parabola would represent the total pressure applied. These pressures, or compressive stresses, are equal and opposite at all mating points of both cylinders.

If a second pair of cylinders of the same size but of softer material, that is, with a lower modulus of elasticity, were in contact under the same pressure as before, the contact surface between them would be broader. As a result, this parabola with a broader base and the same area would have a lesser height. In Fig. 110, this effect is illustrated.

It is not the total pressure or stresses that we are interested in but rather the maximum stress at any points that result from a given load. This stress is known as the "maximum specific stress" and is represented by the height of the parabola shown in Fig. 109 or 110. To determine this maximum specific stress between

two cylinders in contact under load, we apply the Hertz equation, as follows:

$$S^2 = \frac{0.35W\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}{L\left(\frac{1}{E_1} + \frac{1}{E_2}\right)}$$

where S = maximum specific compressive stress, pounds per square inch
W = load on cylinders, pounds
R_1 = radius of first cylinder, inches
R_2 = radius of second cylinder, inches
L = length of cylinder, inches
E_1 = modulus of elasticity of material in first cylinder, pounds per square inch
E_2 = modulus of elasticity of material in second cylinder, pounds per square inches

We can transpose this equation and put it in terms of pitch diameter, pressure angle, and ratio of the pair of gears and thus use it as a measure of the compressive stresses set up in gear teeth. Thus, when

R = radius of curvature of tooth profile at pitch line, inches
α = pressure angle
D = pitch diameter of pinion or smaller gear
$R = \frac{D \sin \alpha}{2}$
f = length of face of gears, inches
Ws = equivalent static tooth load, pounds
K = stress factor $= \frac{S^2 \sin \alpha}{4 \times 0.35}\left(\frac{1}{E_1} + \frac{1}{E_2}\right)$
Q = ratio factor $= \frac{2N}{n + N}$
n = number of teeth in pinion
N = number of teeth in gear

This equation now becomes:

$$Ws = DfKQ \tag{113}$$

The equation was used to compare the data from a large number of different gear drives. This comparison was made by solving for K and thus determining the extent of the maximum

specific stress. Needless to say, the results were far from uniform. This condition is to be expected, however, because all gear drives are not loaded to their extreme capacity. One pertinent fact, however, seemed to stand out: *When the stresses thus determined were below the elastic limit of the material, the wear on such gears was not appreciable. When these stresses exceeded the elastic limit, sometimes the gears stood up, but many of them showed signs of rapid wear. It would seem, therefore, as though the elastic limit of the material were a good stopping point to use in determining safe loads for wear.*

It would seem that the rate of wear should be proportional to the pressures, all other factors being equal. As pointed out by Professor Ham, however, there appears to be a critical point, below which the wear is negligible and above which wear is rapid. When the pressures are below this critical load, it is very possible that the rate of wear is a function of the pressure, velocity, etc., and it is hoped that a series of experiments will soon be underway to determine these relationships. The first step toward this end, however, is to establish the critical load beyond which excessive wear takes place.

As a result of the comparisons made on different gear drives, the writer worked up a set of constants for use in establishing the maximum safe load for wear on gears of different materials. In general, they are based on maximum specific stresses of about 75 per cent of the elastic limit of the weaker material. In order to simplify the equations, a pressure angle of $14\frac{1}{2}$ deg. was assumed as a constant angle. In Table XXVII, these constants are tabulated.

The smoothness of surface plays a very important part in minimizing wear. Table XXVII is based on smooth-cut or ground tooth surfaces with proper lubrication. These values are based on uniform load conditions. Intermittent and shock loads should be studied individually. In all cases, the attempt should be made to determine the tooth loads in terms of an equivalent static load.

One other factor should be mentioned, and that is the relation between the diameter and the width of face of the gear. Sometimes a face of not greater than twice the diameter of the pinion is specified as the maximum face. In general, it would be better to reduce this face to not more than one and one-half times the diameter of the pinion. With very wide faces, the torsional

deflection of the pinion causes the load to be concentrated toward one end, an effect that causes excessive local wear.

TABLE XXVII.—RESULTING SPECIFIC COMPRESSIVE STRESSES WITH STATIC-STRESS FACTORS

Material	Static K $K = \frac{S^2}{5.60}\left(\frac{1}{E_1} + \frac{1}{E_2}\right)$	Maximum specific compressive stress, pounds per square inch	Assumed elastic limit in compression, pounds per square inch
Cast steel and cast steel	25	45,000	60,000
Forged steel and cast steel	30	50,000	80,000 and 60,000
Forged steel and forged steel	40	58,000	80,000
Hardened steel and cast steel[1]	50	65,000	180,000 and 60,000
Forged steel and semisteel	60	55,000	80,000 and 80,000
Hardened steel and bronze[1]	70	62,000	180,000 and 54,000
Hardened steel and semisteel	80	65,000	180,000 and 80,000
Non-metallic and steel	100	23,000	32,000 and 60,000
Semisteel and semisteel	100	58,000	80,000
Heat-treated steel and heat-treated steel	100	92,000	100,000
Hardened steel and heat-treated steel[1]	150	115,000	180,000 and 100,000
Hardened steel and hardened steel	250	145,000	180,000

[1] Experience indicates that the action of a smooth hardened tooth with one of softer material burnishes and hardens the surface of the softer material, thus enabling it to carry heavier loads without wear.

As definite examples, we will use the data from the preceding Examples A and B. Thus, for Example A, we have the following factors:

Equivalent static load = 24,842 (calculated by present practice)
Equivalent static load = 23,662 (calculated by tentative equations deduced from tests on Lewis testing machine)

$$D = 10.000$$
$$f = 6.000$$
$$K = 250 \text{ (hardened steel, from Table XXVII)}$$
$$n = 20$$
$$N = 60$$
$$Q = \frac{2 \times 60}{20 + 60} = 1.50$$

Whence, from Eq. (113), we have

$$W_S = 10 \times 6 \times 250 \times 1.5 = 22{,}500$$

In this example, the safe load for wear is slightly less than the equivalent static load calculated by either present practice or by the tentative equations derived from the results of the tests on the Lewis gear-testing machine. The difference is slight, and, as stated before, these gears have proved to be satisfactory in service.

The safe static beam strength of these gears is equal to 15,573/0.612, which equals 25,446. This value is very close to both the equivalent static load and the safe static wear load and indicates a well-balanced design as regards diameter, width of face, and circular pitch of the gears.

For Example *B*, we have the following factors:

Equivalent static load = 12,748 (calculated by present practice)

Equivalent static load = 8,387 (calculated by tentative equations deduced from tests)

$$D = 7.000$$
$$f = 14.000$$
$$K = 100 \text{ (heat-treated steel, from Table XXVII)}$$
$$n = 21$$
$$N = 84$$
$$Q = \frac{2 \times 84}{21 + 84} = 1.60$$

whence, from Eq. (113), we have

$$W_S = 7 \times 14 \times 100 \times 1.6 = 15{,}680$$

In this example, the safe load for wear is slightly greater than the equivalent static load as calculated by present practice and is nearly double this equivalent static load as calculated by the tentative equations derived from the results of tests on the Lewis machine. As stated before, these gears have also proved satisfactory in service.

The safe static beam strength of these gears is equal to 16,147/0.353, which is equal to 45,884. This value is about three times as great as the wear value and indicates that a finer pitch should be used to obtain a more nearly balanced design between beam strength and wear. This pair would be a better design if the pitch were changed to 6 or 8 d.p. instead of 3 d.p., as actually made.

THE EFFECT OF FILLET RADIUS UPON STRESS DISTRIBUTION

When there is a sudden change in the area of a stressed section, there will be a concentration of stress at or near the place where the sudden change in area takes place. The study of stress distribution and concentration in all forms of sections has been greatly facilitated by the development of the photo-elastic method, by means of which celluloid models are used in conjunction with polarized light so that this problem can be studied visually.

This photo-elastic method has recently been applied to the study of stresses in gear teeth. A paper on this subject by Dr. Paul Heymans and A. L. Kimball, Jr., was presented before the A.S.M.E., in December, 1922. Another paper along similar lines was presented by Dr. S. Timoshenko and R. V. Baud before the American Gear Manufacturers' Association, in May, 1926.

Both of these papers emphasize the influence of the size of the fillet at the root of the gear tooth on the magnitude of the maximum stress concentration developed by a load on the gear tooth. These stresses prove to be appreciably greater than those determined by the use of the usual cantilever beam formula, such as the Lewis formula. Doctor Timoshenko gives the following factors of stress concentration, depending upon the ratio of the radius of the fillet to the thickness of the tooth at the base above the fillet. In order to form a basis for the stress calculation, the unit stress is taken as the stress calculated for the cross-section at the base by the use of the usual cantilever beam formula.

When R = radius of fillet
T = thickness of beam (gear tooth) at base above the fillet

Ratio $\frac{R}{T}$	Factor of Stress Concentration
0.10	2.05
0.27	1.73
0.39	1.49

The ratio 0.10 represents a fillet with a radius equal to 0.157/d.p. on a gear tooth of standard proportions. This table shows that the magnitude of the actual stress concentration because of

the rapid change in the area of the section is over double that given by usual calculations; also, that the strength of a gear tooth can be increased considerably, by means of an increase in the radius of the fillet.

These values are based on experiments with fillets that are true arcs of circles. The actual shape of the fillet on a generated gear, however, is not a true arc of a circle but a generated form of more nearly elliptic or parabolic form where the height is appreciably greater than the width. This modification is such that the change in cross-section is less sudden than when the fillet is a true arc of a circle, and consequently the stress concentration is less. The actual values for such a generated form of fillet should be determined by experiment. It is probable, however, that these values will correspond to those of a fillet with a radius of two to three times the clearance. Even under these conditions, however, the actual maximum stress will be appreciably higher than that given by the cantilever beam formula.

Efficiency of Spur Gears.[1]—The investigations that are briefly discussed in the following pages are characteristic of the work done in this field and have been chosen as the most representative.

In 1886, Wilfred Lewis reported the results of an extensive series of tests made by William Sellers and Company to determine the efficiency of the more common forms of gearing, namely, worm, spiral, and spur. A dynamometer was used to measure the power received by the worm or pinion shaft, and a brake to measure the power delivered to the wheel shaft. These two elements in their proper relation to the gearing to be tested constituted the principal parts of the apparatus employed. In these tests, which were very largely confined to worm gearing, an attempt was made to measure the friction loss between the teeth of spur gears, but the apparatus was not sensitive enough for the purpose, and the errors introduced were therefore relatively excessive. The results obtained indicated a range in efficiency of from 86 to 99 per cent for a good pair of cut spur gears made under average conditions.

Soon after the publication of the Sellers experiments, Prof. J. Burkitt Webb, of the Stevens Institute of Technology, sug-

[1] Abstracted from Ham, C. W., and J. W. Heckert, "An Investigation of the Efficiency and Durability of Spur Gears," *Bull.* 149, Engineering Experiment Station, University of Illinois.

gested to Mr. Lewis the possibility of dividing one of the pair of spur gears to be tested so as to make the load on the teeth self-contained. Accordingly, a testing machine was eventually designed and built that embodied this idea, so that it was possible to run gears under heavy loads at high speeds with a very small consumption of power; this design eliminated brake resistance, which Mr. Lewis had found particularly difficult to stabilize. It was proposed by the Committee on Standards for Involute Gears, of which Mr. Lewis was chairman, to use this apparatus in measuring the friction losses in gear teeth and also in studying the effect of wear.

An investigation with this apparatus was finally undertaken-in 1910, by Messrs. Green and Doble, students in the Massa, chusetts Institute of Technology, as the subject matter for a thesis, under the direction of Prof. Gaetano Lanza. Tests were made to determine friction losses in gear teeth cut to various proportions. So far as is known, the final results of these tests did not appear in the technical press, but from a preliminary report of the tests, which Mr. Lewis had available in 1910, he deduced some values for the friction of gear teeth which may be summed up as follows:

1. 20-deg. stub teeth with addendum about 0.24 of the circular pitch showed less than 1 per cent of friction loss in the teeth.
2. 22½-deg. teeth with addendum about 0.28 of the circular pitch showed about 1 per cent of friction loss.
3. B. & S. 14½-deg. teeth with addendum about 0.32 of the circular pitch showed about 1.3 per cent of friction loss.
4. Bilgram special 15-deg. teeth (long and short addendum, proportions not given) showed over 2 per cent of friction loss.
5. The friction loss in gear teeth is influenced to a greater extent by the length of the addendum than by the angle of obliquity of the system.
6. In no instance is the friction loss of sufficient magnitude to exercise a controlling influence upon the type of gear tooth to be recommended or adopted.

In 1913, the Committee on Standards for Involute Gears presented a report of further experiments on the friction losses in gear teeth. The same apparatus, with some modifications and improvements, had been used, and the tests were conducted at the Massachusetts Institute of Technology by H. S. Waite, under the supervision of Professor Lanza, and in Philadelphia by Everett St. John, at the plant of the Tabor Manufacturing Company, under the direction of Wilfred Lewis. The results of

these tests showed that under ordinary working conditions the friction loss between the teeth of cut gears and pinions seldom exceeds 1 or 2 per cent of the power transmitted; that it is practically independent of the obliquity of action; and that the influence of addendum is the dominant cause of friction. These results confirm the conclusions drawn from the previous experiments of Messrs. Green and Doble.

The machine used in making these tests had certain faults, particularly in regard to the matter of making reliable determinations of the frictional losses in the bearings of the test gears. This situation led to the later and improved design, known as the "Lewis machine," built in 1916 for the University of Illinois and employed in obtaining the results published in this bulletin.

In 1887, Prof. F. Reuleaux published the results of a mathematical investigation relating to the friction of teeth in spur and bevel gearing. Geometrical and analytical investigations of tooth friction for cycloidal and involute gears were made, formulas were developed for the work of friction, and numerical applications of the formulas were given to enable a comparison to be made between the two systems of gearing as regards friction. Theoretical relations between friction and wear of teeth were deduced, and certain conclusions were drawn, one of which was that the work of friction is greater with the involute than with the cycloidal form of tooth.

In discussions of Reuleaux' paper by Wilfred Lewis and others, this conclusion and some of the others drawn by Releaux were challenged on the basis of other methods of computation, which were claimed to be more exact and which led to conclusions precisely opposite to those of Reuleaux.

In 1888, Prof. Gaetano Lanza published the results of a mathematical investigation undertaken for the purpose of throwing light on the conflicting conclusions of Reuleaux, Lewis, and others. Professor Lanza claimed that the conclusions of Reuleaux and also those of his opponents were based on purely theoretical grounds and that all their solutions were approximate. He further asserted that all previous investigations within his knowledge, including those of Rankine, Herrmann, and Mosely, were only approximations; and that, although the reasoning in his own presentation was upon a purely theoretical basis, no approximations had been made in the deduction of formulas for the work lost in friction and the consequent efficiency of the

gears when the friction of the journals in their bearings was not taken into account.

Professor Lanza then discussed the assumptions on which his formulas were based and, after deducing formulas for the efficiency of gears, gave numerical efficiencies that he had computed for certain cases of cycloidal and involute teeth. The following is a brief summary of his conclusions:

1. Whether the cycloidal or involute tooth is the more efficient depends upon the proportions used for each.
2. The efficiency of involute gears is not, as claimed by G. B. Grant and others, independent of the obliquity.
3. The differences in certain efficiencies (examples referred to in his paper), as computed from the formulas, are so small that they could easily be masked by several indeterminate factors.
4. The question of efficiency can be correctly answered only by experiment.

In an extended discussion of Lanza's paper by Prof. J. Burkitt Webb, Wilfred Lewis, and others, some decided exceptions were taken to his mathematical treatment of the subject, and the general theories pertaining to friction and wear of gear teeth were analyzed and discussed at length.

A study of the history of gear testing shows that other investigations have found the determination of the frictional losses in gear teeth elusive and difficult to repeat with precision. The slightest variation in any one of the many factors involved obviously will have an appreciable effect on such small quantity as tooth-friction loss. In this investigation, similar difficulties were encountered, and although the Lewis machine was sufficiently sensitive to justify the conclusions drawn, it is believed that a specially constructed machine of extreme accuracy and sensitiveness is necessary to register the effect of some of the minor factors that may affect the frictional losses in gear teeth. The following is a summary of the author's conclusions on efficiency:

1. The efficiency of unhardened gears is practically independent of the quantity of oil used for lubrication, provided the quantity is sufficient to prevent heating and cutting.
2. The efficiency is independent of the speed within the range covered by this investigation, namely, a pitch-line speed of 60 to 1,500 ft. per minute.
3. The efficiency does not appear to be influenced by the obliquity of action.

4. For all practical purposes, the efficiency is independent of the load transmitted. The value of 99 per cent is suggested for use in computations dealing with the efficiency of gears cut in accordance with good commercial practice.

5. Condition of tooth surface is the most important of the factors that affect the efficiency of unhardened gears. Gears with rough tooth surfaces are less efficient than those in which the tooth surfaces have become glazed, but the difference in efficiency is not so great as has been commonly assumed.

6. When all other conditions are the same, greater sliding action causes the longer addendum gears to have a slightly lower efficiency than the shorter addendum gears. On the other hand, the vibration of the longer addendum gears may, for certain ratios, be so much less than that of the shorter addendum gears, as to result in a slightly higher efficiency of the long addendum gears.

7. The difference in efficiency of the several standard tooth forms in common use is so small as to exercise no controlling influence on the tooth form to be recommended or adopted for any purpose.

SECTION III

MACHINING AND MEASURING GEAR TEETH

CHAPTER IX

MEASURING GEAR TEETH

The problem of gear production is essentially the same as all other production problems. The requirements of the finished product should be determined first, and then suitable methods must be selected or developed to produce the required results. Often, the exact requirements of the part or the product in question are not fully established until actual production has been underway for some time. This situation imposes on production the additional burden of unanticipated refinements in the manufacturing operations. Such, in brief, has been the history of the production of almost every commodity.

The history of gear production has been no exception to this but rather an aggravated example. In the first place, the essential requirements of gear teeth have not yet been definitely established. Secondly, the demands made on the performance of gears are constantly growing more severe. In the third place, although many different methods for producing gears are now available, not one of them has yet proved itself to excel in all features; each method has its own advantages, and all have certain disadvantages. The first essential to satisfactory production is knowledge of the essential requirements or elements. If the essential element of a composite surface, such as a gear-tooth profile, is singled out for primary attention, a very high order of accuracy can be obtained. The fewer elements we try to maintain to any specified degree of accuracy at one time, the higher the degree of accuracy it is possible to maintain.

The next essential is suitable means of measuring. We must be able to detect and measure definitely any troublesome errors before we are in a position to correct them. Here, again, we are at a disadvantage in the production of gears. Many valuable and ingenious testing instruments have been devised, but, almost without exception, they are more of the type of laboratory instruments than everyday shop testing tools. Such laboratory instruments have an important place to fill, but production requires

testing facilities that are simple, rapid, and effective. Until we know definitely the characteristics of the gear teeth that require continual watching, however, we are at a disadvantage when we attempt to evolve facilities to test them. This is the principal reason why such testing means have not yet made their appearance. Despite the lack of this essential—suitable means of measuring—production must be maintained. In fact, all production is carried on despite more or less inadequate information and equipment. In general, measurement or inspection should be employed more as a preventative than as a cure. Inspection methods that will prevent faulty parts from being produced are of far more value than those that merely sort the good parts from the bad. The ideal inspection and production system would be to have tools and equipment of such reliability that periodic tests of this equipment alone would insure correct results. This is the end toward which we should always strive. The more nearly any production process approaches this condition, the less the attention that is required to measure the product. The measurement of the tools and the machines used to produce the gears should always receive primary attention. Proper inspection here will save the production of faulty parts. The inspection of this equipment should be much more complete and elaborate than the measurement of the gears themselves. Too often, however, inspection is considered only as a process of sorting the product. Some sorting is always necessary, but this is a minor and not a major duty of inspection.

Fundamentally, the measurement of gears is the same as the measurement of any other mechanical part. With gear-tooth profiles, as with all other types of surfaces, we are confronted with the fact that the accuracy to which we can work depends upon the accuracy to which we can measure. As stated before, we must be able to measure errors before we can correct them.

Before we can measure and make intelligent use of such measurements, however, three questions must be answered: First, what shall we measure? Second, why should we measure it? And third, how shall we measure it?

For most machined surfaces, these questions have relatively simple answers. Gear-tooth profiles, however, have so many interdependent elements, that satisfactory answers have not as yet been found for all of them. The further discussion will single out the more important elements of gear-tooth profiles

and show typical methods that are now employed to measure them.

The distinction between the testing of the manufacturing equipment and the product itself should always be kept in mind. That is, measurements made to test the accuracy of the production equipment can be as elaborate as necessary with but little effect on the final cost of the product, because such tests are required periodically only. If simple methods, therefore, are not available, the more elaborate ones should be used. On the other hand, measurements that are made to test the product itself must be simple and direct to be economical. When considering the measurement of gears, therefore, and the various types of measuring equipment available, these two distinct purposes should not be confused. We will now consider the more essential characteristics of gear-tooth profiles. Spur-gear teeth of full-involute form only will be considered. The more essential elements of such gears may be listed as follows: thickness of teeth; concentricity of teeth; alignment of teeth; spacing of teeth; and form of tooth profiles.

Why Should the Thickness of Gear Teeth Be Measured?—If the teeth are too thick, the gears cannot be assembled at the desired center distance. If the teeth are too thin, excessive backlash will be present. In extreme cases, the teeth would be weakened.

The actual amount of backlash required depends upon several factors. The more accurately the center distance is maintained, the smaller the amount of backlash required. As a matter of fact, the tolerance on the center distance for spur gears should always be plus, as a variation in that direction introduces only a little more backlash, while a variation in the opposite direction reduces the backlash and introduces the possibility of jamming the teeth. The amount of backlash required depends also upon the pitch-line speed of the gears. High-speed gears require more backlash than slower ones, in order to prevent jamming when the gears are heated in operation. On the other hand, for gears running at moderate speeds, the smaller the backlash the less chance is there of rattling when the gears are running idle. This factor is of particular importance on the constant-mesh gears of automobile transmissions.

After the minimum backlash is established, the maximum is controlled largely by the process of manufacturing. The

tolerance on the tooth thickness would vary from about 0.001 inch on ground gears to about 0.005 inch on cut gears of 5 d.p. and finer On coarser pitches, this variation will be somewhat larger.

In general, the measurement of the tooth thickness of gears is the simplest method of control in production. If the other elements of the gear teeth can be satisfactorily maintained by periodic tests of the equipment and by proper care in the set-up of the blanks, the measurement of tooth thickness alone on the product would be sufficient.

How Can the Thickness of Gear Teeth Be Measured?—Several methods are available for measuring the thickness of gear teeth.

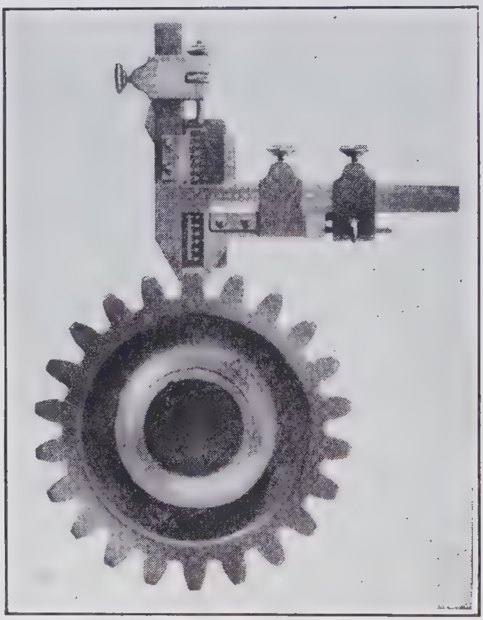


Fig. 111.—Gear-tooth vernier. The horizontal scale measures the chordal thickness, the vertical one, the distance of this chord from the outside diameter.

Some of these methods measure the thickness directly, while others measure it indirectly and also give an indication of the errors in some other elements at the same time.

First, we have the gear-tooth vernier, which is shown in Fig. 111. This instrument has two vernier scales, one to measure the chordal thickness of the gear teeth, and the second to measure the distance from this chord to the outside diameter of the gear.

If this outside diameter varies, a corresponding correction must be made in the setting of the second vernier scale.

Calculating the Tooth Thickness.—Usually, the arc thickness of the tooth is known. In order to measure with this instrument, it is necessary to calculate the chordal thickness and also the distance of this chord from the outside diameter of the gear. Referring to Fig. 112, when

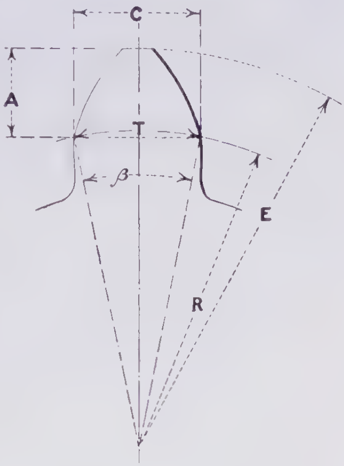


Fig. 112.—Application of the gear-tooth vernier.

T = arc thickness of gear tooth at radius R, inches
R = radius at which arc thickness of tooth is known, inches
E = outside radius of gear, inches
C = chordal thickness of gear teeth, inches
A = distance from outside of gear to chord (known as "corrected addendum"), inches
β = angular thickness of gear tooth

$$\text{arc } \beta = \frac{T}{R} \tag{114}$$

$$C = 2R \sin \frac{\beta}{2} \tag{115}$$

$$A = E - R \cos \frac{\beta}{2} \tag{116}$$

As a definite example, we will determine the value of the chordal thickness and corrected addendum on the following gear tooth:

$$T = 0.260 \text{ in.}$$
$$R = 1.000 \text{ in.}$$
$$E = 1.1667 \text{ in.}$$
$$\text{arc } \beta = \frac{T}{R} = \frac{0.260}{1.000} = 0.260 \text{ radian}$$
$$\beta = 14 \text{ deg. } 54 \text{ min.}$$
$$C = 2R \sin \frac{\beta}{2} = 2 \sin 7 \text{ deg. } 27 \text{ min.} = 0.2593 \text{ in.}$$
$$A = E - R \cos \frac{\beta}{2} = 1.1667 - \cos 7 \text{ deg. } 27 \text{ min.} = 0.1751 \text{ in.}$$

The actual contact between this gear-tooth vernier and the gear tooth is at the corner of the jaws. Hence, if this corner becomes worn, the instrument will give a false reading. If accurate measurements are to be obtained, these instruments should be calibrated from time to time and the necessary correction made to the reading of the instrument.

A simple method of calibrating these gear-tooth verniers is shown in Fig. 113. The theoretical addendum and chordal-thickness settings may be readily calculated for any diameter of

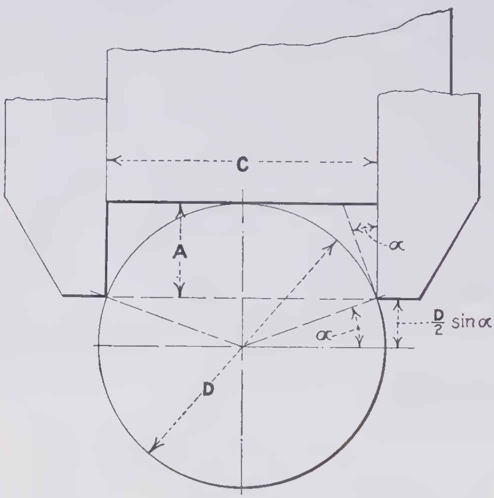


Fig. 113.—Calibration of a gear-tooth vernier.

cylindrical plug gage that may be available. It is a good plan to calibrate these instruments for the same angle as the pressure angle of the gears to be measured. Thus, when

D = diameter of plug gage, inches
A = addendum setting or height of segment, inches
C = thickness setting or length of chord, inches
α = pressure angle of gears to be measured, degrees

$$A = \frac{D}{2}(1 - \sin\alpha) \tag{117}$$

$$C = D\cos\alpha \tag{118}$$

To calibrate the instrument, the height vernier is set to the calculated addendum setting A, and the length of the chord C is measured with the other vernier. The difference between this

measurement and the calculated length of the chord C is the correction to be used when measuring the thickness of gear teeth.

As a definite example, we will calculate the settings for calibration purposes on a 1-in. plug gage for a pressure angle of 20 deg. This gives us the following:

$$D = 1.000 \text{ in.}$$
$$\alpha = 20 \text{ deg.}$$
$$A = \frac{D}{2}(1 - \sin \alpha) = \frac{1}{2}(0.65798) = 0.3290 \text{ in.}$$
$$C = D \cos \alpha = 0.9397 \text{ in.}$$

If the actual measurement of the chord should be 0.9380 in., we must add the correction of 0.0017 in. to the reading of the vernier when measuring a gear tooth to obtain the correct measurement.

These instruments cover a wide range of pitches and pressure angles, thus making them particularly valuable for use on general jobbing work. The smallest one covers a range from 20 to 2 d.p., while a larger instrument covers the range of from 10 to 1 d.p.

Another instrument which is used to measure the tooth thickness of gears is the gear-tooth micrometer, which is shown in Fig. 114. The base of this instrument represents a space of the basic rack and fits over the tooth of the gear to be measured, while the micrometer is adjusted to touch the outside diameter of the gear. A variation in the height thus measured indicates a variation in the thickness of the tooth. If the outside diameter of the gear varies, a corresponding correction must be made in the reading of this instrument.

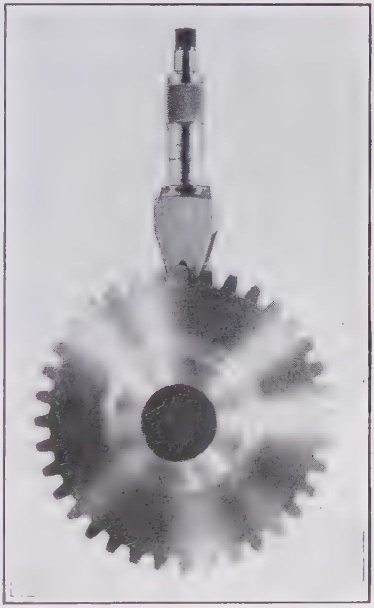


FIG. 114.—Gear-tooth micrometer measuring the contact depth of the rack-shaped opening. The width of the tooth is then calculated.

This instrument makes contact with the gear-tooth profile on a plane surface. A separate instrument is required for each different pitch. The same instrument may be used on gears of different pressure angles, but this involves considerable calculation, so that it is usually advisable to have a separate instrument for each pitch and pressure angle. These instruments are primarily intended for extensive production gaging.

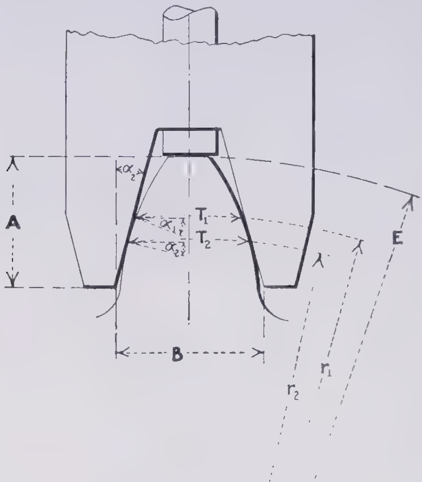


FIG. 115.—Use of the gear-tooth micrometer.

When one of these gear-tooth micrometers of one pressure angle is used to measure a gear tooth of different pressure angle, the following calculation must be made. Referring to Fig. 115, when

A = gage or micrometer setting, inches
T_1 = arc thickness of gear tooth at r_1, inches
r_1 = pitch radius of gear with pressure angle of α_1, inches
α_1 = pressure angle of *gear*, degrees
T_2 = arc thickness of tooth at r_2, inches
r_2 = pitch radius of gear for pressure angle of α_2, inches
α_2 = pressure angle of *gage*, degrees
B = width at bottom of space in gage, inches
E = outside radius of gear, inches

We have, from Prob. 6 in Chap. III,

$$r_2 = \frac{r_1 \cos \alpha_1}{\cos \alpha_2} \qquad \text{(see Eq. (39))}$$

$$T_2 = 2r_2\left(\frac{T_1}{2r_1} + \operatorname{inv} \alpha_1 - \operatorname{inv} \alpha_2\right) \qquad \text{(see Eq. (40))}$$

From Fig. 115, we have

$$A = E - r_2 + \frac{B - T_2}{2} \cot \alpha_2 \tag{119}$$

As a definite example, we will take a 20-deg., 6-d.p. gear with 24 teeth, tooth thickness 0.260 in., and calculate the gage setting for a $14\frac{1}{2}$-deg., 6-d.p. gear-tooth micrometer. These instruments are usually made so that the gage setting represents the working depth of the tooth on a gear tooth of basic size and pressure angle; whence,

$$B = \frac{CP}{2} + \frac{2}{6}\tan 14\tfrac{1}{2}\text{ deg.} = 0.3480\text{ in.}$$

$$\begin{aligned} T_1 &= 0.2600\text{ in.} \\ r_1 &= 2.0000\text{ in.} \\ \alpha_1 &= 20\text{ deg.} \\ \alpha_2 &= 14\tfrac{1}{2}\text{ deg.} \\ B &= 0.3480\text{ in.} \\ E &= 2.1667\text{ in.} \end{aligned}$$

$$r_2 = \frac{r_1 \cos\alpha_1}{\cos\alpha_2} = \frac{2 \times 0.93969}{0.96815} = 1.9412\text{ in.}$$

$$\begin{aligned} T_2 &= 2r_2\left(\frac{T_1}{2r_1} + \text{inv }\alpha_1 - \text{inv }\alpha_2\right) \\ &= 3.8824\left(\frac{0.2600}{4.0000} + 0.01490 - 0.00554\right) = 0.2887\text{ in.} \end{aligned}$$

$$\begin{aligned} A &= E - r_2 + \frac{B - T_2}{2}\cot\alpha_2 \\ &= 2.1667 - 1.9412 + \frac{0.3480 - 0.2887}{2}\cot 20\text{ deg.} \\ &= 0.3070\text{ in.} \end{aligned}$$

Roll Measurement of Gear-tooth Thickness.—The measurement of the thickness of gear teeth by means of rolls placed between the teeth, as shown in Fig. 116, has several advantages. In the first place, these measurements are independent of the outside diameter of the gear. In the second place, this method does not require the provision of any special measuring instruments. On the other hand, the determination of the correct distance over the top of the rolls involves considerable calculation, and for this reason, the method of measuring over rolls has not been so extensively used as it otherwise would be. Oftentimes, rolls of a diameter that will permit them to contact at the pitch line of the gear are provided. This practice has been followed because the calculations involved under these conditions are relatively simple and obvious, although it requires

the provision of rolls of special diameter. This restriction is not necessary, however, as rolls of any convenient size may be used. The calculation for the measurement over the rolls is made as follows: Referring to Prob. 13 in Chap. III, we have

r_1 = radius at which tooth thickness is known
α_1 = pressure angle at r_1
T_1 = tooth thickness at r_1

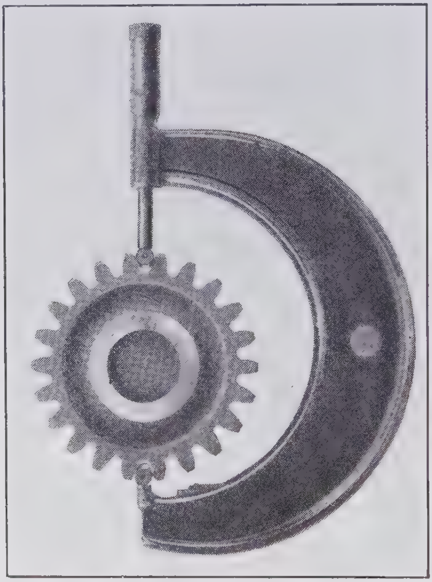


FIG. 116.—Roll measurement of gear-tooth thickness.

W = radius of roll
r_2 = radius from center of gear to center of roll
α_2 = pressure angle at r_2
N = number of teeth in gear

$$\text{inv } \alpha_2 = \frac{T_1}{2r_1} + \text{inv } \alpha_1 + \frac{W}{r_1 \cos \alpha_1} - \frac{\pi}{N} \qquad \text{(see Eq. (53))}$$

$$r_2 = \frac{r_1 \cos \alpha_1}{\cos \alpha_2} \qquad \text{(see Eq. (54))}$$

When the number of teeth in the gear is even, so that the center line of two opposite spaces passes through the center of the gear,

$$M = \text{measurement over rolls} = 2(r_2 + W) \qquad (120)$$

When the number of teeth in the gear is odd, so that a space is opposite a tooth, a correction must be made for this misalignment. In this case,

$$M = 2\left(r_2 \cos \frac{90\text{ deg.}}{N} + W\right) \tag{121}$$

Table XXVIII gives values of $\cos \frac{90\text{ deg.}}{N}$ for odd numbers of teeth from 5 to 99, to assist in these calculations. As a definite example, we will determine the measurement over 0.375-in. diameter rolls on a 25-tooth, 5-d.p., 20-deg. gear with a tooth thickness of 0.3120 in. on its pitch line. This gives us the following:

$$\begin{aligned} r_1 &= 2.5000 \text{ in.} \\ \alpha_1 &= 20 \text{ deg.} \\ T_1 &= 0.3120 \text{ in.} \\ W &= 0.1875 \text{ in.} \\ N &= 25 \text{ teeth} \end{aligned}$$

From Eq. (53), we have

$$\text{inv } \alpha_2 = \frac{0.3120}{5.0000} + 0.01490 + \frac{0.1875}{2.500 \times 0.93969} - \frac{3.1416}{25} = 0.03145$$

whence, $\alpha_2 = 25$ deg. 23 min. and $\cos \alpha_2 = 0.90346$

From Eq. (54), we have

$$r_2 = \frac{2.500 \times 0.93969}{0.90346} = 2.6002 \text{ in.}$$

$$M = 2\left(r_2 \cos \frac{90}{N} + W\right) = 2(2.6002 \times 0.99803 + 0.1875) = 5.5652 \text{ in.}$$

At times, it is necessary to calculate the thickness of the teeth from the measurement over the rolls. This merely reverses the preceding calculations. Using the same symbols as before, we have, when the number of teeth in the gear is even,

$$r_2 = \frac{M - 2W}{2} \tag{122}$$

When the number of teeth in the gear is odd, we have

$$r_2 = \frac{M - 2W}{2 \cos \frac{90\text{ deg.}}{N}} \tag{123}$$

N = D.P. = ------

ROLL MEASUREMENT OF TOOTH THICKNESS

When tooth thickness is known				When roll measurement is known			
$r_1 =$		When N is even $M = 2(r_2 + W)$		$r_1 =$		$\cos\alpha_2 = \frac{r_1 \cos\alpha_1}{r_2}$	
$\alpha_1 =$				$\alpha_1 =$			
$T_1 =$		$r_2 =$		$W =$		$\cos\alpha_1 =$	
$W =$		$W =$	+	$M =$		$r_1 \cos\alpha_1 =$	
$\text{inv}\,\alpha_2 = \frac{T_1}{2r_1} + \text{inv}\,\alpha_1 + \frac{W}{r_1 \cos\alpha_1} - \frac{\pi}{N}$		Sum =		When N is even $r_2 = \frac{M - 2W}{2}$		$\cos\alpha_2 =$	
		$M =$				$\alpha_2 =$	
$\cos\alpha_1 =$		When N is odd $M = 2\left(r_2 \cos\frac{90°}{N} + W\right)$				$T_1 = 2r_1\left(\frac{\pi}{N} + \text{inv}\,\alpha_2 - \text{inv}\,\alpha_1 - \frac{W}{r_1 \cos\alpha_1}\right)$	
$r_1 \cos\alpha_1 =$				$M =$			
$T_1/2r_1 =$		$\cos\frac{90°}{N} =$		$2W =$		$\frac{3.1416}{N} =$	
$\text{inv}\,\alpha_1 =$	+	$r_2 \cos\frac{90°}{N} =$		Diff. =		$\text{inv}\,\alpha_2 =$	+
$\frac{W}{r_1 \cos\alpha_1}$	+	$W =$	+	$r_2 =$		Sum =	
Sum =		Sum =		When N is odd $r_2 = \frac{M - 2W}{2\cos\frac{90°}{N}}$		$\text{inv}\,\alpha_1 =$	−
$\frac{3.1416}{N}$		$M =$				Diff. =	
$\text{inv}\,\alpha_2 =$		*r_1 = Radius at which tooth thickness is desired* *α_1 = Pressure angle at r_1* *T_1 = Tooth thickness at r_1* *W = Radius at roll* *r_2 = Radius to center of roll* *α_2 = Pressure angle at r_2* *M = Measurement over rolls*				$\frac{W}{r_1 \cos\alpha_1} =$	−
$\alpha_2 =$				$\cos\frac{90°}{N} =$		Diff. =	
$\cos\alpha_2 =$				$2\cos\frac{90°}{N} =$		$2r_1 =$	
$r_2 = \frac{r_1 \cos\alpha_1}{\cos\alpha_2}$				$M =$		$T_1 =$	
				$2W =$			
$r_2 =$				Diff. =			
				$r_2 =$			

FIG. 117.—Form for use in calculating the tooth thickness by roll measurement.

Transposing Eq. (54), we have

$$\cos \alpha_2 = \frac{r_1 \cos \alpha_1}{r_2} \tag{124}$$

Transposing Eq. (53), we have

$$T_1 = 2r_1\left(\frac{\pi}{N} + \operatorname{inv} \alpha_2 - \operatorname{inv} \alpha_1 - \frac{W}{r_1 \cos \alpha}\right) \tag{125}$$

If many of these calculations are to be made, the work is very much simplified by the use of a form. Such a form, suitable for a 5 by 8-in. card is shown in Fig. 117.

Another simple method[1] of measuring the thickness of involute gear teeth, that employs standard measuring instruments and is not affected by variations in the outside diameter of the gear blank, consists of measuring across two or more teeth, depending upon the pressure angle and number of teeth in the gear, with standard vernier calipers. In Fig. 118 this method is shown.

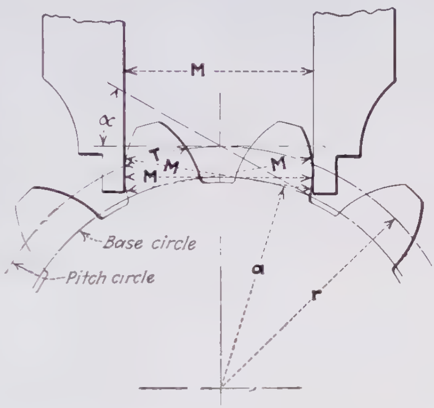


FIG. 118.—Measurement of tooth thickness by vernier caliper.

The dimension measured is equal to the length of the circumference of the base circle between the two points where the two outside involute profiles rise from the base circle. This distance is exactly the same in any direction normal to the two involute profiles.

The determination of the correct distance across these two profiles involves a simple calculation. This may be done as follows:

When M = measurement across profiles, inches
r = pitch radius of gear, inches
α = pressure angle of gear, degrees
T = tooth thickness at radius r, inches
N = number of teeth in gear
S = number of tooth spaces between measured profiles

$$M = r \cos \alpha\left(\frac{T}{r} + \frac{\pi S}{N} + 2 \operatorname{inv} \alpha\right) \tag{126}$$

[1] WILDHABER, ERNEST, *American Machinist*, vol. 59, p. 551.

It is desirable to measure these profiles somewhere near the pitch line of the gears, because this portion of the profile is usually free from modification and also because the pitch-line area is an important part of the active profile. Thus, as the number of teeth in the gear increases, the base circle is at a greater distance below the pitch circle, so that it is necessary to include more tooth spaces between the measured teeth in order to keep the contact between the jaws of the calipers and the tooth profile near the pitch-line area. Table XXIX gives the values of S to use on gears of different tooth numbers and different pressure angles.

TABLE XXVIII.—ROLL MEASUREMENT OF GEARS WITH ODD NUMBER OF TEETH

N	$\frac{90 \text{ deg.}}{N}$	$\cos \frac{90 \text{ deg.}}{N}$	N	$\frac{90 \text{ deg.}}{N}$	$\cos \frac{90 \text{ deg.}}{N}$
			51	1° 45′ 53″	0.99953
			53	1° 41′ 53″	0.99956
5	18° 0′ 0″	0.95106	55	1° 38′ 11″	0.99959
7	12° 51′ 26″	0.97493	57	1° 34′ 44″	0.99962
9	10° 0′ 0″	0.98481	59	1° 31′ 32″	0.99965
11	8° 10′ 54″	0.98982	61	1° 28′ 31″	0.99967
13	6° 55′ 23″	0.99271	63	1° 25′ 43″	0.99969
15	6° 0′ 0″	0.99452	65	1° 23′ 5″	0.99971
17	5° 17′ 39″	0.99573	67	1° 20′ 36″	0.99973
19	4° 44′ 13″	0.99658	69	1° 18′ 16″	0.99974
21	4° 17′ 9″	0.99720	71	1° 16′ 3″	0.99976
23	3° 54′ 47″	0.99767	73	1° 13′ 58″	0.99977
25	3° 36′ 0″	0.99803	75	1° 12′ 0″	0.99978
27	3° 20′ 0″	0.99831	77	1° 10′ 8″	0.99979
29	3° 6′ 12″	0.99853	79	1° 8′ 21″	0.99980
31	2° 54′ 12″	0.99872	81	1° 6′ 40″	0.99981
33	2° 43′ 38″	0.99887	83	1° 5′ 4″	0.99982
35	2° 34′ 17″	0.99899	85	1° 3′ 32″	0.99983
37	2° 25′ 57″	0.99910	87	1° 2′ 4″	0.99984
39	2° 18′ 28″	0.99919	89	1° 0′ 10″	0.99984
41	2° 11′ 42″	0.99927	91	0° 59′ 20″	0.99985
43	2° 5′ 35″	0.99933	93	0° 58′ 4″	0.99986
45	2° 0′ 0″	0.99939	95	0° 56′ 51″	0.99986
47	1° 54′ 54″	0.99944	97	0° 55′ 40″	0.99987
49	1° 50′ 12″	0.99949	99	0° 54′ 33″	0.99987

TABLE XXIX.—TOOTH SPACES BETWEEN TEETH MEASURED BY VERNIER CALIPER

S =	1		2		3		4		5		6		7		8	
Pressure angle of gear, degrees	Number of teeth in gear															
	From	To	From	To	From	To	From	To	From	To	From	To	From	To	From	To
14½	12	25	19	37	38	50	51	62	63	75	76	87	88	100		
17	12	21	22	32	33	42	43	53	54	64	65	74	75	85	86	96
20	12	18	19	27	28	36	37	45	46	54	55	63	64	72	73	81
22½	12	16	17	24	25	32	33	40	41	48	49	56	57	64	65	72
25	12	14	15	21	22	29	30	36	37	43	44	51	52	58	59	65

As a definite example, we will calculate the vernier-caliper measurement for a 6-d.p., 20-deg. gear with 30 teeth and a tooth thickness on the pitch line of 0.2600 in. This gives us the following values:

$$r = 2.500 \text{ in.}$$
$$\alpha = 20 \text{ deg.}$$
$$T = 0.260 \text{ in.}$$
$$N = 30 \text{ teeth}$$

From Table XXIX, we get, for a 20-deg., 30-tooth gear,

$$S = 3$$

whence,

$$M = 2.500 \cos 20 \text{ deg.}\left(\frac{0.260}{2.500} + \frac{3\pi}{30} + 2 \text{ inv. } 20 \text{ deg.}\right) = 1.0524 \text{ in.}$$

The difference between the measured value and the calculated value gives directly the amount that the teeth are thick or thin.

The tooth thickness of gears may be measured indirectly by the measurement of the root diameter, as shown in Fig. 119.

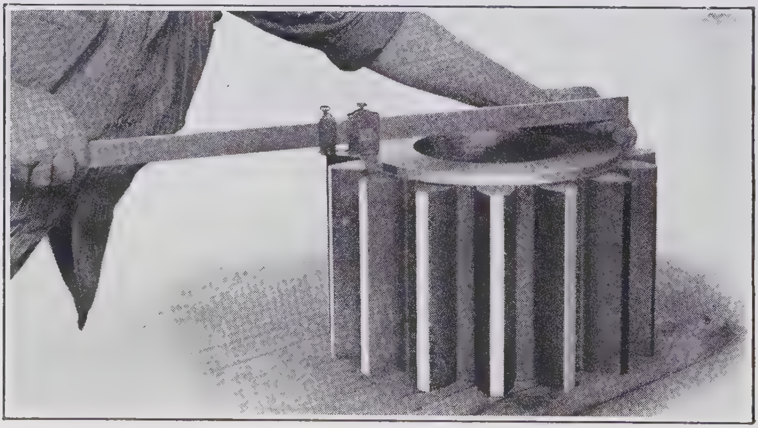


FIG. 119.—Measurement of root diameter.

This illustration shows the measurement of a gear with an even number of teeth. When the number of teeth in the gear is odd, the measurement is made from the outside diameter of one tooth to the bottom or root of the opposite space. When the tooth dimensions of the generating tools are accurately maintained, this method gives, in many cases, a sufficiently accurate measure

of the tooth thickness. For larger gears, in particular, where considerable backlash is permissible, it makes a simple and effective method for the control of the tooth thickness.

In addition to the foregoing methods of measuring the tooth thickness independently, there are several other tests that give an indication of tooth thickness in combination with an indication of the accuracy of other elements. One of such methods consists of meshing a pair of gears at the correct center distance. These methods will be considered later with other composite or functional tests. For the present, we will consider only those methods that test each single element individually.

Why Should the Concentricity of Gear-teeth Profiles Be Measured? Suitable precautions should be taken in the manufacture of the gear blanks and in their set-up on the gear-cutting machine so that it is not necessary to test the finished product for eccentricity. In most cases, a small amount of eccentricity is of minor importance. The tooth action between eccentric involute gears is smooth and continuous, causing a fluctuation in velocity that follows an almost pure sine curve.

Eccentricity is extremely objectionable, however, on change gears used for accurate dividing or screw cutting. Here, even small eccentricities have a marked effect on the accuracy of the final result. This is due to the cumulative effect of such errors on the relative positions of the cutting tools and the work. Particular pains should always be taken in the cutting of change gears to keep the amount of eccentricity to an absolute minimum.

Tests for concentricity should be tests of the set-up rather than inspection or sorting operations on the product itself. In other words, any test of the product itself is used primarily as a check on the set-up.

How Should the Eccentricity of Gear-tooth Profiles Be Measured? Several methods are available to measure the eccentricity of the tooth profiles, both direct and indirect. For the present, we will consider only the direct measurement of the eccentricity.

One of the simplest and most commonly used methods of testing the eccentricity of gears employs a cylindrical pin or roll and an indicator. This method may be applied in many ways.

For gears of small size, the gear is mounted on centers, the pin is placed in a tooth space and rolled under a dial indicator. The pin is then moved to the next space and the operation is repeated. After the position of the pin in all of the tooth spaces has been

thus tested, the difference between the largest and smallest reading on the dial indicator is the amount of run-out. The eccentricity is one-half of this amount. In Fig. 120, this method is illustrated.

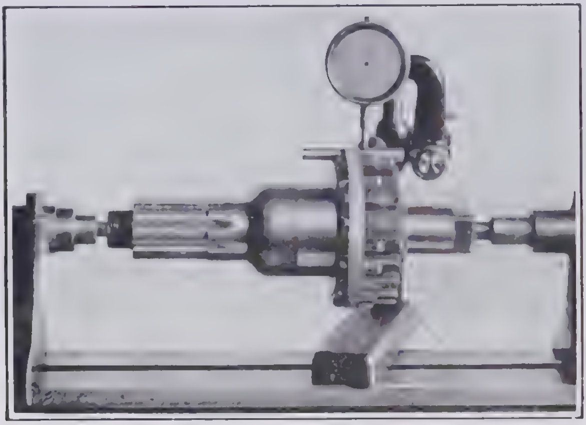


FIG. 120.—Measuring the eccentricity of a small gear.

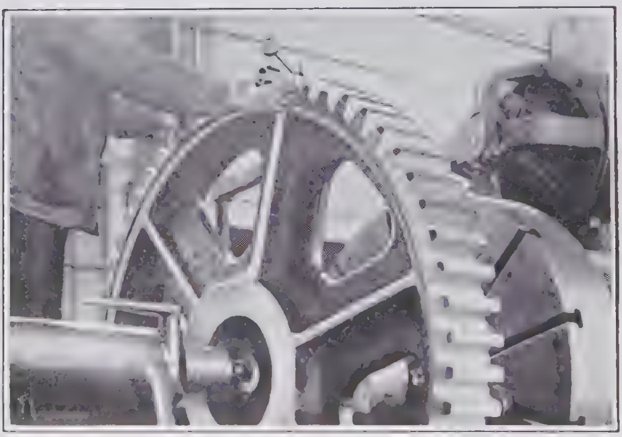


FIG. 121.—Measuring the eccentricity of a large gear mounted in a lathe.

This same method is shown in Fig. 121, applied to a larger gear. In this case, the gear is mounted in a large lathe. The same method can also be applied, particularly on very large gears, while the gear is in the cutting machine. In this case, however, the arbor on which the gear is centered or the bore of the gear, if

it is not carried on an arbor, should also be tested for concentricity at the same time.

Another modification of this method is shown in Fig. 122. In this case, the cylindrical pin is extended to carry a bent rod on which a dial indicator is mounted. An arbor or bushing is placed in the bore of the gear, and the indicator is swung past the bushing with the pin held in the tooth space as a pivot.

Another method of measuring the eccentricity of gear-tooth profiles consists of mounting the gear in a testing machine and meshing it tightly with a master gear, as shown in Fig. 123. One of the arbors on which the gears are mounted is stationary,

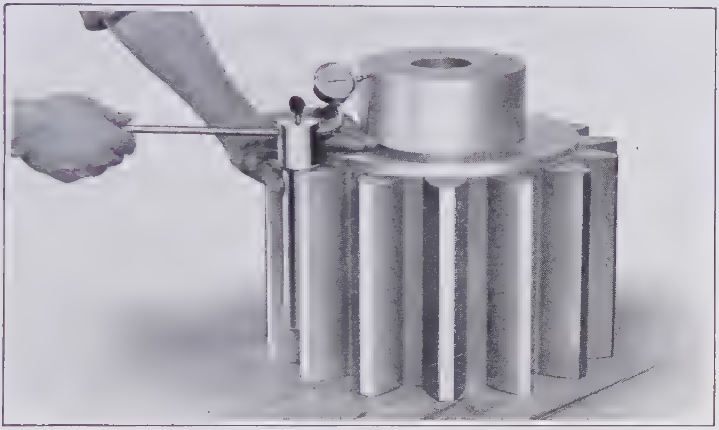


FIG. 122.—Another method of checking the eccentricity.

while the other is mounted on a slide. The gears to be tested are held tightly together by the pressure of a spring acting against the movable slide. As the gears are rotated, any eccentricity that may be present causes the slide to move back and forward accordingly. The dial indicator at the end of the machine is used to measure any movement of this slide. The difference between the largest and smallest reading on the dial indicator shows the amount of run-out of the gear-tooth profiles. As noted before, this run-out is double the eccentricity.

Another method of measuring the eccentricity of gear-tooth profiles employs a projection lantern. In this case, the gear is mounted on a stud, while a tooth profile that is from 90 to 120 deg. away from the profile being projected is held against a stop. A line representing the tooth profile is drawn on the screen, and

the stop is adjusted so that this line coincides with the projection of the first tooth tested. Successive profiles are then brought in contact with the stop. Any deviations of the successive projected profiles from the line on the screen are measured. In effect, this measures the cumulative spacing errors over the number of teeth between the stop and the profile projected on the screen. Such cumulative errors are primarily the results of eccentricity, so that this method is practically a direct measure of eccentricity, and the measured deviations on the screen can be converted into actual gear errors.

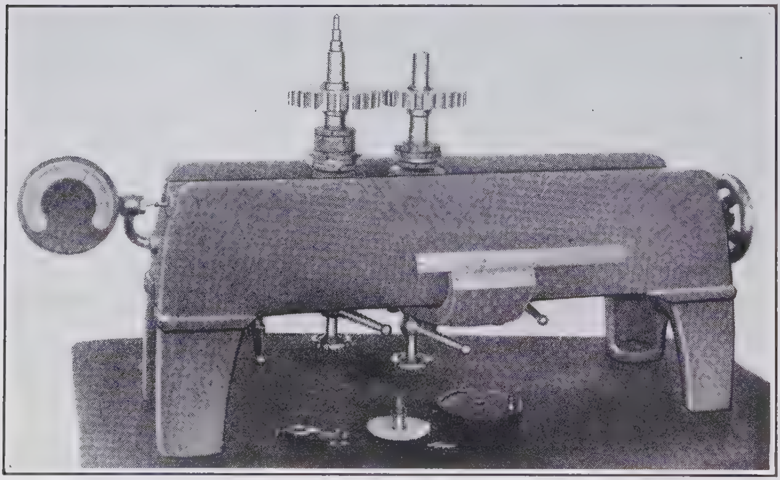


FIG. 123.—Machine for testing eccentricity of a pair of gears.

Why Should the Alignment of Gear Teeth Be Measured?—If mating gear-tooth profiles are not in almost perfect alignment, the load will not be distributed evenly across the face of the gear. If the misalignment is appreciable, the load will be concentrated at one edge of the gear blank, thus causing excessive local wear as well as excessive noise at high pitch-line velocities.

As with eccentricity, the alignment of the gear teeth should be maintained by care in setting up the blanks for machining as well as care in maintaining the gear-cutting machines themselves in good operating condition.

How Can the Alignment of Gear Teeth Be Measured?—The simplest and most effective method of testing the alignment of the gear teeth consists of mounting the mating pair of gears at

their correct-center distance on parallel shafts and running them together under a sufficient load to show the nature of the contact between them. Gears thus tested should show contact entirely across the face.

The alignment of the teeth of large spur gears may be tested by setting them up on a surface plate and testing both the bore and the tooth profiles with a try-square.

Why Should the Spacing of Gear Teeth Be Measured?—If the spacing of the gear teeth is not uniform, the gears in operation will be noisy and also the maximum load carried by the teeth will be increased because of such errors, or conversely the permissible load to be carried will be correspondingly reduced.

The accuracy of the spacing of gear teeth is controlled primarily by the accuracy of the gear-cutting machine and the accuracy of the generating tools. Measurements of spacing on a cut gear, therefore, should be made to test the accuracy of the production equipment rather than the product itself. If tests of the product show spacing errors of troublesome amounts, both the generating tools and the gear-cutting machine should be tested to locate the source of the trouble. Suitable steps should then be taken to correct whatever is at fault. Excessive spacing errors on the product should always be a signal to stop production and correct the equipment.

How Should the Spacing of Gear Teeth Be Measured?—Several methods are available to test the spacing of gear teeth. One simple method is to place two pins in adjacent tooth spaces and to measure over them, as shown in Fig. 124. As these pins are moved from space to space, any differences in these measurements indicate a variation in the spacing of approximately the measured difference.

This same method, applied in a slightly different way, is shown in Fig. 125. In this case, the pins are extended, and a bent rod carrying a dial indicator is mounted in one of them. This is the same device as is shown in Fig. 122. The indicator is rocked past the second pin, and any differences in the readings as the pins are moved from space to space indicate the errors in spacing.

The odontometer is a simple and self-contained instrument for testing the accuracy or uniformity of the gear-tooth spacing. The instrument illustrated in Fig. 126 has a range of from 3 to 10 d.p. and may be used to check gears of any pressure angle. It can be applied to a gear while it is in place in the machine.

Fig. 124.—Checking spacing errors by pins.

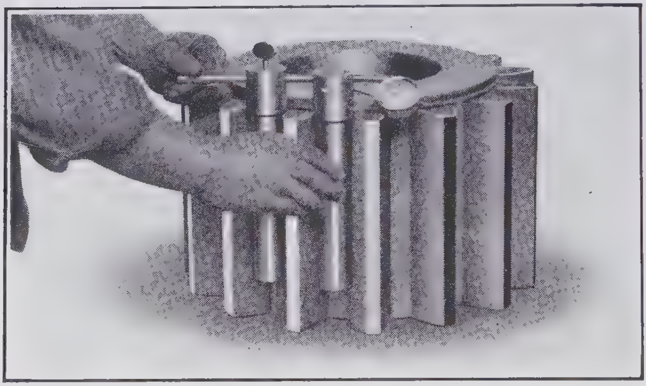


Fig. 125.—Modified method with indicator.

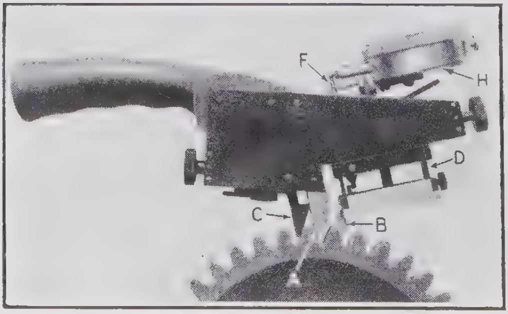


Fig. 126.—Odontometer.

In effect, it is composed of a section of a straight-sided rack with two parallel effective faces, one fixed and the other movable. A third face, set at an angle to the two working faces, is used to hold the fixed working face in contact with the flank of the gear tooth. The fixed registering surface is at *A*; the movable indicating surface, at *B*. The third surface *C* holds surface *A* in contact with the involute surface of the gear tooth. The surfaces *B* and *C* are adjustable, so that gears of various pitches can be tested with the same instrument.

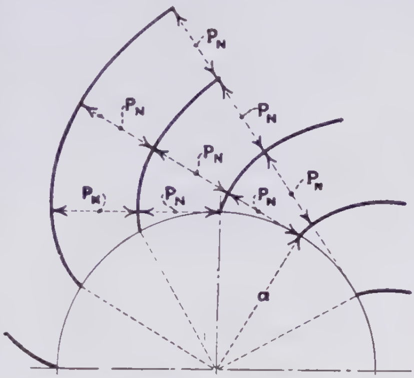


FIG. 127.—Family of involute curves.

The indicating surface *B* is mounted on two, thin, flat springs *D*, which act as pivots free from backlash. The dial indicator *H* is actuated by the lever *F*, which has a ratio of 5:1, so that each division on the dial represents a movement of 0.0002 in. of the indicating surface *B*. In order to explain the operation of this instrument, it is necessary to reconsider some of the characteristics of the involute profile. Figure 127 shows a series of involutes equally spaced on a given base circle. The normal pitch *Pn* between these involutes along a line tangent to the base circle is always the same, no matter where the tangent is drawn. Figure 128 shows a few gear teeth and indicates the portion of the involute where contact is made with the odontometer. The instrument is located over one tooth and rocked into contact with the adjacent one. When the movable face of the instrument reaches position *M*, the distance *Pn* should be registered. With true involute profiles, the hand on the dial will remain still until the instrument reaches position *N*. In operation, this momentary pausing of the hand on the dial is taken as the reading.

FIG. 128.—Theory of odontometer measurements.

In general, the instrument is used as a comparator to test the uniformity of spacing of interchangeable and mating gears. This normal pitch *Pn*, as noted previously, should be the same on mating gears. If actual measurements are required, the distance between the two parallel working faces of the instrument can be measured. Figure 129 shows this measurement being made with size blocks.

A larger instrument with a range of from ¾ to 4 d.p. is shown in Fig. 130. This instrument has two dial indicators, one on either side, so that it can be used with either *side* of the instrument up when held in a horizontal position while testing large gears which are still set up on the cutting machine.

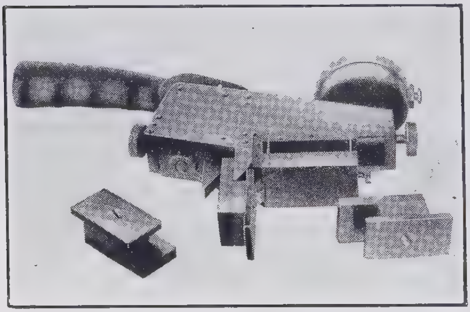


Fig. 129.—Checking the odontometer reading with size blocks.

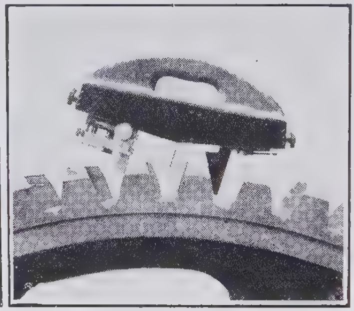


Fig. 130.—Odontometer for large gears.

A somewhat similar instrument, designed by A. W. Copland, is shown in Fig. 131. This is more of a single-purpose instrument for production checking and requires a separate one for each different pitch. It consists of a roll that fits into a tooth space, and thus makes contact on one tooth profile, and a pivoted finger that makes contact with the next tooth profile. Any variation in the position of this finger is recorded on the dial indicator. An adjustable stop is also provided so as to control the position where the finger makes contact on the tooth profile.

The Lees-Bradner gear tester may be adapted to test the spacing of gear teeth, as shown in Fig. 132. The gear, in this case, is left free to rotate upon its arbor while the tooth-spacing fixture and the slide frame are clamped firmly in position. The adjustable stop J is set to engage one tooth while the contact point of the lever K engages the next tooth. The long end of this lever engages the dial indicator M, which is usually set at zero

on the first pair of teeth tested. The gear is then lifted out of engagement with the stop J, and the contact point of the lever K is moved one tooth and again brought against the stop. The indicator reads plus or minus variations from the original setting, as each successive tooth is brought into contact with the stop J. These readings are in tenths of thousandths of an inch and indicate a variation in tooth spacing of practically the full amount indicated.

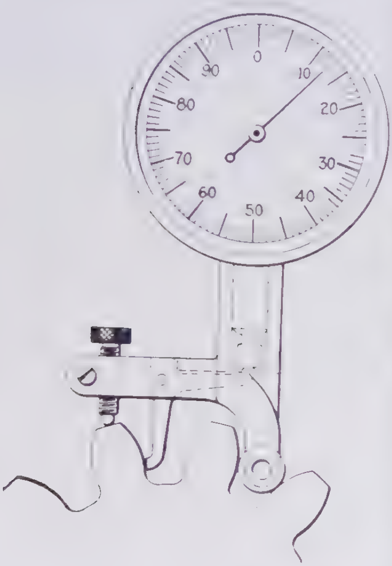


FIG. 131.—Copland tooth-space measuring instrument.

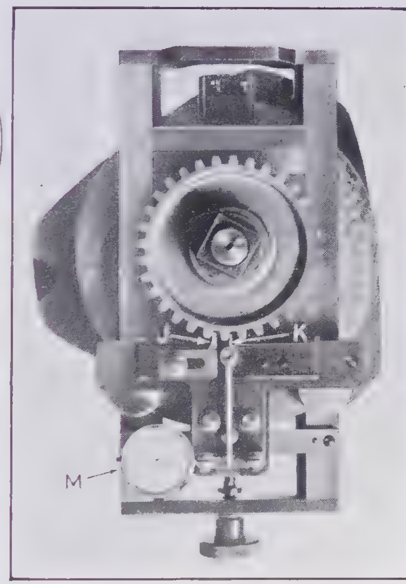


FIG. 132.—Lees-Bradner spacing tester.

Why Should the Form of Tooth Profiles Be Measured?—An involute gear is a series of uniform-rise cams, used to transmit power or motion. If the transmitted motion is to be smooth and uniform, both the cam profiles and their relation to each other must be correct; otherwise, irregular and noisy action will result.

Considerable stress is often laid upon the necessity of easing off the profile of involute gear teeth. Assuming rigid teeth of perfect form and spacing, such modifications are unnecessary. But as these conditions are never fully attained in practice, such a procedure often proves advisable. Edge contact at the beginning of mesh is particularly troublesome and should always be

avoided. The amount of modification required is only just enough to avoid such edge contact and, thus, depends entirely upon the accuracy of the gears. The more closely they approach perfection, the less modification they require. Too much modification is often worse than none at all. The effect of such modification is to reduce the amount of contact. The safe, permissible amount of modification is limited by the length of contact between the given pair of gears. If the "overlap" is small, very little modification can be permitted if quietly running gears are to be secured. Gear teeth designed to give longer lengths of contact offer greater opportunities of suitable modifications in order to correct for small errors, and, hence, smoother and quieter action can be secured.

Although, theoretically, the entire involute profile is suitable for use as a gear-tooth profile, that part of it immediately above the base circle is so sensitive that it can be neither measured nor produced accurately. Many times, gears whose profiles and spacing seem to be almost perfect are far from satisfactory in operation. In the majority of such cases, it will be found that such gears attempt to use the involute profile almost to its origin at the base circle, and errors are present there that cannot be detected by direct measurement. The best practice is to eliminate this portion of the involute form from the active profile of the gear teeth. This can be accomplished in one of three ways: The first and most satisfactory method is to design the gear-tooth forms so that the active profile stops an appreciable distance away from the base circle; about $\frac{1}{32}$ in. on 5-d.p. gears, for example. This is accomplished by extending the addendum of the pinion with a corresponding reduction in the addendum of the gear.

The second method is to remove entirely this portion of the profile. This can be accomplished by designing the generating tools so that they undercut and remove this part of the profile or by introducing a separate milling operation to remove this part of the profile. The third method is to ease off or modify the tip of the profile of the mating gear tooth so that it will not contact with the involute profile near the base circle.

The accuracy of gear-tooth profiles depends primarily upon the accuracy of the gear-cutting machines and the cutting tools used to produce them. The accuracy of these profiles can be maintained only by a rigid control of the accuracy of the production

equipment. Here, also, the testing of the profiles of the product should be used as a check on the accuracy of the equipment rather than as a product-inspection operation. The production of inaccurate gear-tooth profiles should therefore be a signal to stop the production, test the equipment, and correct the trouble at its source.

How Shall the Gear-tooth Profiles Be Measured?—Several instruments are now on the market for testing the accuracy of involute profiles. In most cases, they consist of a disk representing the

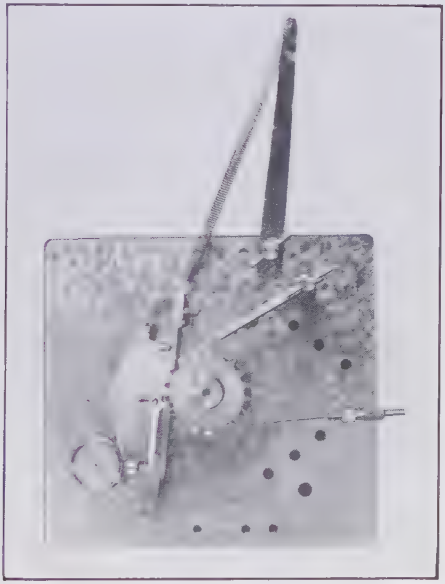


Fig. 133.—Kavle indicator.

base circle of the involute to be measured, a straight-edge constrained to rock on this base circle disk without slipping, and an indicator point so located as to represent the end of the line that is unwound from the base circle to generate the involute profile. If the pointer of the indicator remains stationary while the point travels over the tooth profile, it indicates that the form of the tooth is of the correct involute shape. If it moves, the direction and amount of its movement gives a measure of the error in the tooth profile.

One instrument of this type is the Kavle indicator, shown in Fig. 133. It consists of a base plate, a disk of the same diameter

as the base circle of the gear to be inspected, a stud or bushing on which the gear is located, and a straight-edge which is held in contact with the base-circle disk by means of steel ribbons, 0.002 in. in thickness, and a spring. On the straight-edge is mounted a 5:1 lever, the short end of which is in contact with the tooth form while the long arm is in contact with the plunger of the dial indicator. This 5:1 lever multiplies the motion so that each division on the dial is equivalent to 0.0002 in. As the straight-edge is rolled on the base-circle disk, the point on the short arm follows the tooth form of the gear. If the tooth profile is of true

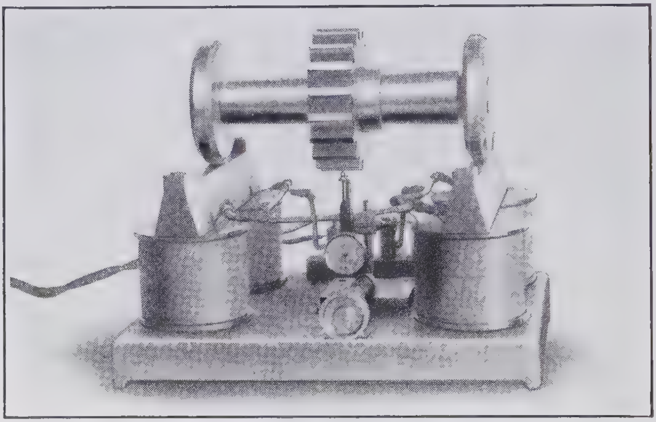


FIG. 134.—Involute-form tester.

involute form, the pointer of the indicator will not move. If errors are present, the dial indicator will show the direction and amount. The accuracy of the spacing of the teeth may also be measured by this same instrument by bringing each successive tooth in contact with a stop and taking the dial reading when the bar is held in a fixed position.

Another instrument of this type is shown in Fig. 134. This consists of a pair of straight-edges on which roll two base-circle disks of equal diameter. A magnetic circuit is made through the straight-edge and disk to prevent slippage. The gear to be tested is mounted on the shaft between the two base-circle disks. An indicator is mounted on the base of the instrument, the point of the finger of which is located so that it represents the end of the line which is unwound from the base circle to form the involute

curve. The operation of this device is identical to the one previously discussed.

The Lees-Bradner gear tester, shown in Figs. 135 and 136, is another instrument for testing the accuracy of involute profiles, but it is developed in greater detail than either of the preceding ones. The base of this testing machine carries the base-circle disk A with provision for holding the straight-edge B in contact with the base-circle disk with sufficient pressure to insure that it will rock around this disk without slipping. The contact lever

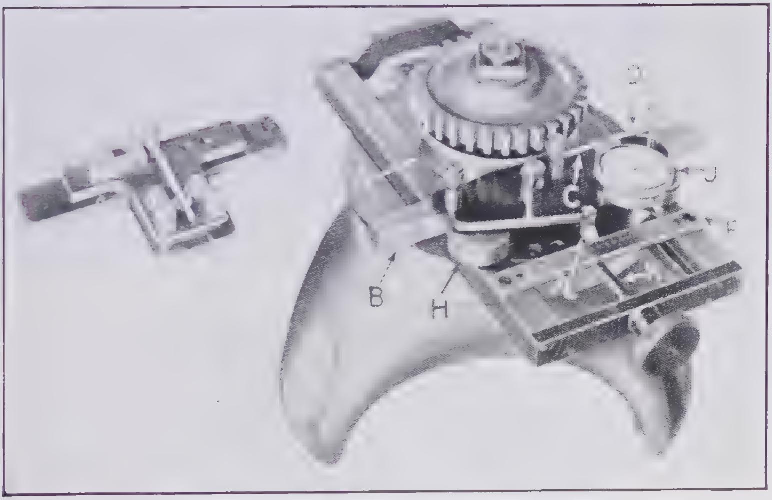


FIG. 135.—Lees-Bradner involute-profile tester.

C is mounted upon the straight-edge with its point in line with the side of the straight-edge that forms the tangent to the base-circle disk. The two rolls H are mounted upon ball bearings carried by the sliding frame and are held against the straight-edge by spring pressure, which can be adjusted by the tension screw K. The entire frame is pivoted upon the arbor that holds the base-circle disk. When the frame is moved about this center, the straight-edge is caused to rock upon the base-circle disk, which remains stationary, and the point of the lever C traces the involute curve of this base circle.

Any movement of the lever C away from the true involute curve is magnified by means of the lever D and caused to register on the dial indicator J, each division of which indicates a move-

ment of the point of the lever C of 0.0001 in. A line on the block L, which is carried by the straight-edge, indicates upon the graduated scale F the distance that the straight-edge has been rocked around the base-circle disk. This distance, divided by the radius of the base-circle disk, gives the angle in radians that the straight-edge has been rocked. It is thus possible to determine when the contact point of the lever C reaches certain successive positions on the involute curve.

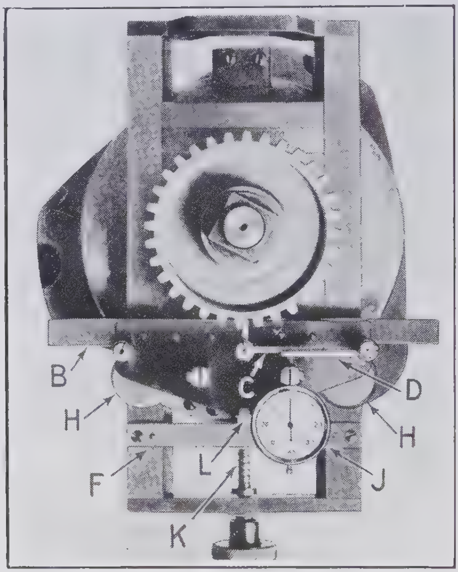


Fig. 136.—Plan view of the Lees-Bradner profile tester.

Figure 136 shows a gear in position for the testing of its tooth profile with the point of the lever C engaging one tooth of the gear, which is clamped over the base-circle disk by means of the nut on the upper part of the arbor. The dial indicator is usually set at zero when the pointer touches the tooth profile at the base circle or when the pointer touches the tooth profile at the bottom of the working depth when this point is outside the base circle. The line in block L coincides with zero on the scale F when the point of the lever is at the base circle. When the teeth are not deep enough to reach the base circle, the readings will start a few divisions beyond the zero. The frame is now moved to

rock the straight-edge upon the base circle, causing the contact point of the lever to travel along the profile of the gear tooth.

The graduations on the scale F mark off a number of successive positions of the point of the lever C as it travels up along the profile of the gear tooth, so that a record may be kept of the indicator readings at each of the scale graduations. This record may be plotted, and the resulting graph not only makes a very convenient manner in which to keep the records but also gives a graphic picture of the form of the profile. The chart shown in Fig. 137 provides a very valuable record of this sort.

This chart provides a place to keep the record of both the profile and the spacing of any gears. The curved lines on the profile chart are approximations to the involute form, while the vertical lines represent equal angular divisions along this involute profile. A formula is given and a place is provided to record the angular motion of the straight-edge in degrees and decimals of a degree for each of the divisions on the scale F. This enables the records of the indicator readings to be readily plotted on this chart. The distance between adjacent curved lines may represent any arbitrary amount, usually 0.0001 in., as in the example plotted. After the tooth profiles of a mating pair of gears have been tested and recorded on these charts, it is a simple matter to solve graphically for many of the contact conditions, such as duration of contact, height of active profile, etc. To do this, the pitch line is first established on this chart. If the gears are to run, for example, at a pressure angle of 20 deg., this pitch line will be at the 20-deg. vertical line on the chart. The indicator on the testing machine will fall off rapidly when the top of the tooth profile has been reached. This enables the position of the top of the active profile to be established on the chart. This will be somewhere between the last two readings made. If, for example, this position comes at the 35-deg. line on the chart, it tells us that there will be 15 deg. of action on the addendum of this gear against the dedendum of its mating gear. This enables us to establish the position of the bottom of the active profile on the chart of its mating gear. If the mating gears are of equal size, there will be the same angular action on the dedendum as on the addendum, and the bottom of the active profile would be at the 5-deg. line on the chart.

If the gears are not of equal size, the angular distance on the dedendum of the mating gear will be inversely proportional

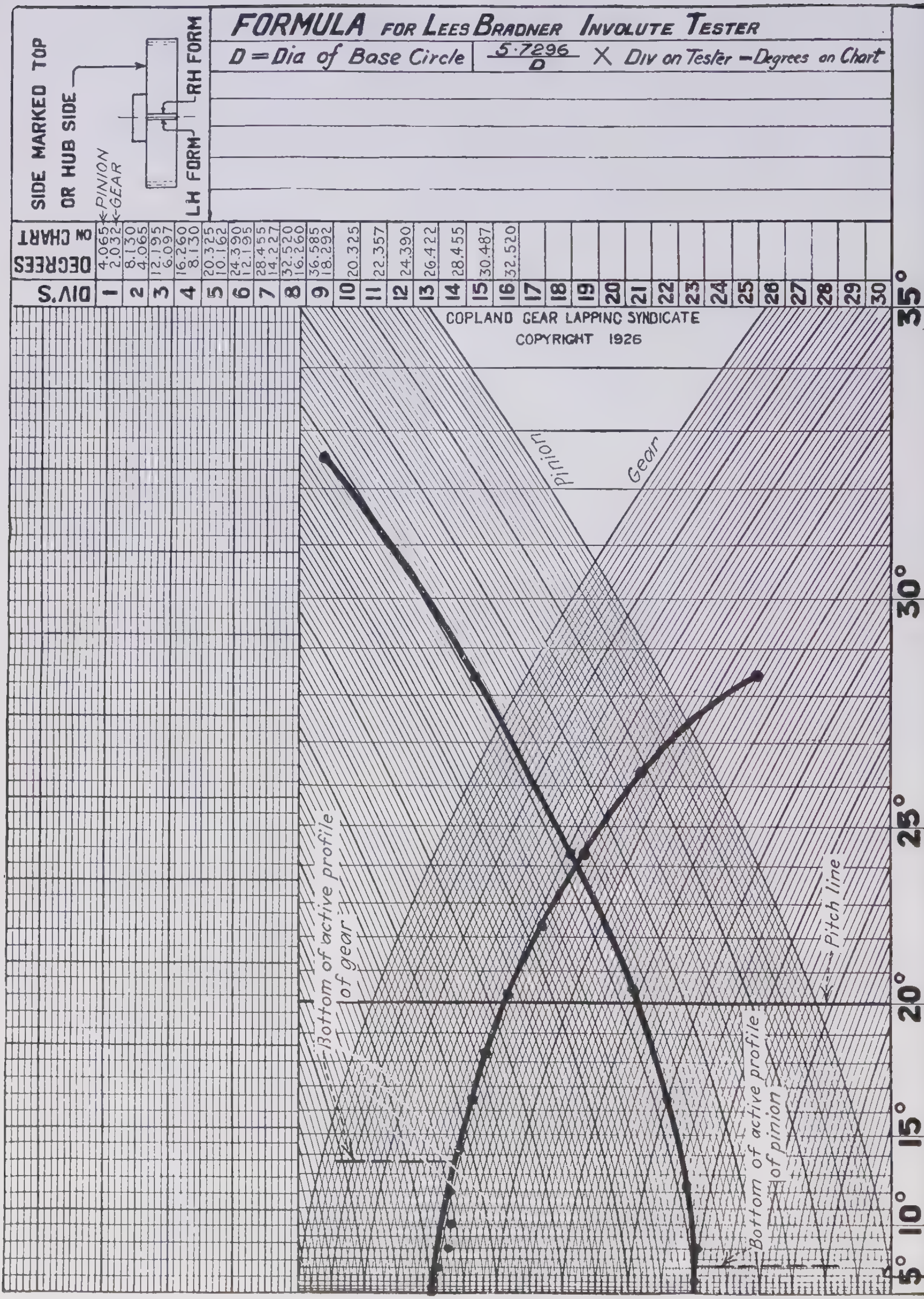


FIG. 137.—Copland chart for use with Lees-Bradner profile tester.

to the ratios of the tooth numbers. Thus, if the mating gear is twice as large as the first gear, this position will be one-half of 15 deg., or $7\frac{1}{2}$ deg. below the pitch line, or at the $12\frac{1}{2}$-deg. line on the chart of the mating gear. On the other hand, if the first gear had 30 teeth and the second gear 20 teeth, this position will be $\frac{30}{20}$ of 15 deg. or $22\frac{1}{2}$ deg. below the pitch line. As this would bring it below the base circle, where involute contact cannot take place, it would indicate that the tooth design developed interference. In like manner, the top of the active profile of the second gear would be established, which would enable the position of the bottom of the active profile of the first gear to be located.

The angular distance between the top and the bottom of the active profile on the chart gives directly the angle through which contact takes place. If this angle should be 25 deg., and the number of teeth in the gear is 18, making angular tooth intervals of 20 deg., this means that contact exists through $\frac{25}{20}$, or 1.25 tooth intervals.

The radius of the pitch circle and the radius of the base circle are always known. The difference between these two radii is represented on the chart by the distance between the 0-deg. line and the pitch line. This enables the linear scale of the chart in the direction of the height of the tooth to be readily determined. Thus, for example, if the difference between the pitch radius and the base-circle radius is 0.095 in., and the distance on the chart to the pitch line is 3.250 in., the chart gives a magnification of the tooth profile in height of 3.250/0.095 or of 34.2 times the actual size. Knowing this scale of enlargement, it is a simple matter to determine the height of the active profile.

A little study of this chart will make apparent how sensitive the involute form is near the base circle. The angular divisions are very close together at this point. Regardless of the actual shape of the tooth on this part of the profile, the indicator on the testing machine will register no movement for the first two or three divisions on the scale, because the pointer will be in contact with the highest adjacent point on the tooth profile and remain in contact with it during the motion of the straight-edge through the first few divisions. Any readings on this part of the involute profile will always be questionable.

As a definite example, we will plot on this chart the mating profiles of an 18-tooth, 20-deg., 6-d.p., full-depth tooth pinion

meshing with a 36-tooth gear. The following tabulation represents the readings on the testing machine:

Division	18-tooth pinion	36-tooth gear
1	0.	0.
2	+0.0001	−0.000
3	+0.0001	−0.0001
4	+0.0002	−0.0002
5	+0.0003	−0.0002
6	+0.0002	−0.0001
7	+0.0001	−0.0001
8	−0.0002	0.
9	−0.0020	+0.0001
10	....	+0.0001
11	....	−0.0001
12	....	−0.0003
13	....	−0.0005
14	....	−0.0015

For the pinion, one division on the testing machine is equal to 4.065 deg. on the chart. For the gear, one division is equal to 2.032 deg. on the chart. The figures for the pinion are written on the chart above those for the gear.

The tip of the tooth of the pinion reaches about the $32\frac{1}{2}$-deg. line, or $12\frac{1}{2}$ deg. above the pitch line, which is a 20 deg. As the ratio is 2:1, the bottom of the active profile of the gear will therefore be $6\frac{1}{2}$ deg. below the pitch line. The tip of the tooth of the gear is about 7 deg. above the pitch line, so that the bottom of the active profile of the pinion will be about 14 deg. below the pitch line. Thus, the active profile of the pinion covers $26\frac{1}{2}$ deg. of action, while that of the gear covers one-half of this amount, or $13\frac{1}{4}$ deg. One tooth interval on the pinion is equal to $^{360}/_{18}$ or 20 deg. Thus, the duration of contact is equal to $26\frac{1}{2}/20$ or 1.325 tooth intervals.

Projection apparatus, similar to that developed for the inspection of screw threads, is sometimes used for the inspection of gear-tooth profiles. The enlarged image or shadow of the gear-tooth profile is thrown on a screen and compared with an enlarged outline of the correct profile, which is drawn there. The enlargement employed is usually about 100:1. One type of such an

apparatus is shown in Fig. 138. In this case, to avoid optical difficulties encountered in trying to get a clear and accurate shadow from a surface of considerable width, a series of needles with their points in the optical plane are pressed against the

FIG. 138.—Projection method of testing tooth profiles.

tooth profile so that the shadow of the points of these needles defines the tooth outline. This method is described in detail by Ralph E. Flanders, in a paper presented before the American Gear Manufacturers' Association, in April, 1922.

FIG. 139.—Instrument for measuring a tooth profile at various points.

Another type of profile-testing equipment is shown in Fig. 139. This instrument consists of a base with a slide that carries the gear. A micrometer head is used to locate the top of the tooth, while two dial indicators are arranged with two levers to measure the tooth thickness at various heights. The gear is shifted from

position to position, and the readings on the dial indicators are recorded. After these measurements are taken, they are plotted as coordinates to an enlarged scale. This type of instrument is often used to get a record of the amount of wear on the teeth of gears. A pair of gears is measured, then placed in a testing machine and run under load for a certain length of time, then removed and measured again. The successive measurements are then plotted on the same chart, which thus shows graphically the effects of wear.

Why Should Gears Be Given Composite Tests?—It is not the effect of errors on individual elements of the tooth profiles that causes uneven and noisy operation so much as it is the cumulative effect of all of such errors. Such cumulative effects can be determined only by means of some composite test.

At times, an error in one element tends to compensate for the effect of an error in some other element. At other times, the resultant is more nearly the sum of the individual errors. Given an infinite variety of product and infinite time and patience, it would be possible to mate up smooth-running pairs regardless of the nature of the profile errors present. But this is not a satisfactory production process.

Composite tests on gears are for the purpose of measuring the smoothness of action of a pair of mating gears, regardless of the errors that may be present on the individual elements of the tooth form. The attempt should be made to establish suitable tolerances on all of the several elements so that the composite tests will also prove satisfactory. Until such tolerances are established, it will prove a very difficult task to exercise any great degree of control over the production operations.

One of the simplest composite tests, that is very widely used, consists of mounting a pair of mating gears at the correct center distance and of rolling them together by hand. For this purpose, several machines of similar design are on the market. Such a machine is shown in Fig. 140. These testing machines consist of a bed with one fixed spindle and one movable spindle. A scale is usually mounted on the bed that may be read to .001 inch by means of a vernier caliper carried on the sliding head. These spindles are set at a predetermined distance apart, then the pair of gears to be tested is mounted on the spindles.

Gears are tested on centers for three purposes: first, to insure that they will run at the proper center distance with suitable

backlash; second, to test them for concentricity; and third, to test the smoothness of the action of the teeth.

The amount of backlash may be measured by inserting thin feeler gages between the engaging teeth or by inserting pieces of paper until the gears mesh tightly and then measuring the thickness of the paper inserted. After gears have been assembled in their mechanism, the backlash is sometimes measured by inserting a piece of lead or solder wire between the teeth and forcing it through the meshing point of the gears, thus flattening the wire. This flattened wire is then measured to

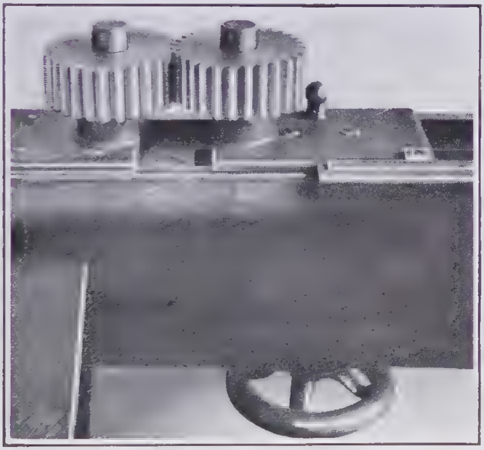


Fig. 140.—Machine for making a composite test of gears.

determine the amount of backlash that is present. If the gears are eccentric, a varying amount of backlash will be found at different positions of the gears.

Another method of measuring the backlash is to adjust the spindles of the machine until the gears are meshed tightly together. The difference between this center distance and the correct one multiplied by twice the tangent of the pressure angle is equal to the amount of backlash.

To test the smoothness of action between the gear teeth, the gears are rolled together by hand, the driven gear being held back while the driving gear is rolled ahead. Under these conditions, the gears should roll together without any feeling of teeth, that is, they should feel the same as two plain disks rolled together.

With practice, operators become very skillful in detecting improper tooth action by feel in this simple test.

Sometimes accurate master gears are provided for use on these testing machines, and the gears are tested against these masters. Again, this same type of test is also often used to select pairs of gears to run together.

The use of the testing machine shown in Fig. 140 is limited to gears of relatively small size. A similar test is sometimes given to larger gears by providing means to hold the pinion in engagement with its mating gear before this gear is removed from the

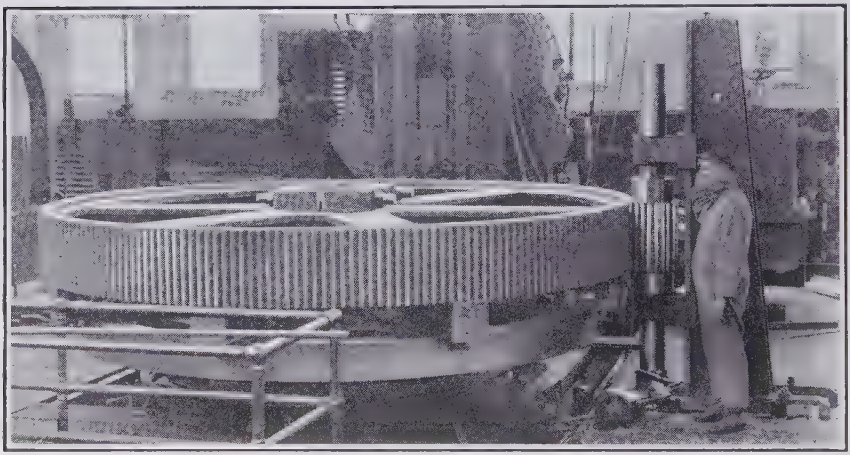


FIG. 141.—Testing a large gear with its mating pinion.

gear-cutting machine. In Fig. 141, such an arrangement is shown.

A similar but more elaborate composite testing machine than those just described indicates or measures the uniformity and smoothness of action between a pair of gears instead of depending upon the "feel" of the operator. The Saurer testing machine, shown in Fig. 142, is a machine of this type.

A diagram of the operating parts is shown in Fig. 143. The gears are mounted on arbors spaced the correct distance apart. On each arbor is also mounted a plain disk, accurately ground to the pitch diameter of the gear. On one arbor, both the gear and the plain disk are mounted on the same sleeve. On the other arbor, which also carries the indicating device, the gear is mounted on an external sleeve, and the pitch disk on an internal sleeve. The

indicator is fastened to the same sleeve that carries the pitch disk, while an arm that engages with the indicator is attached to the external sleeve that carries the gear. A plate which holds

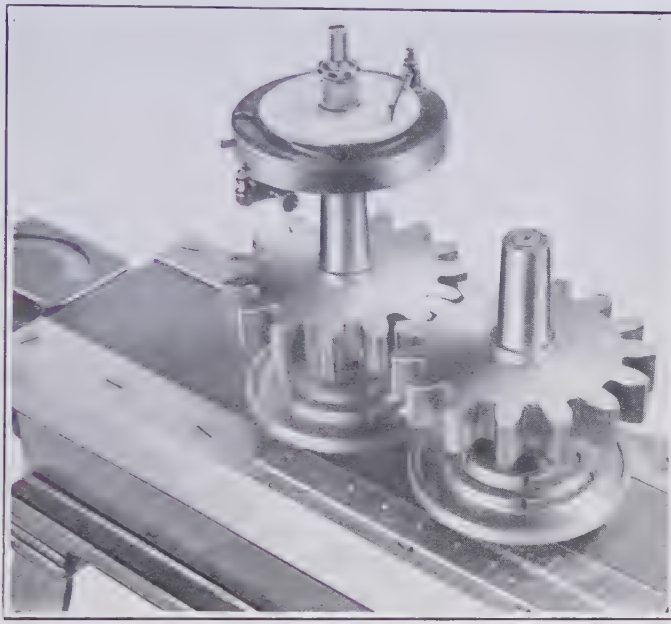


Fig. 142.—Saurer composite gear-testing machine.

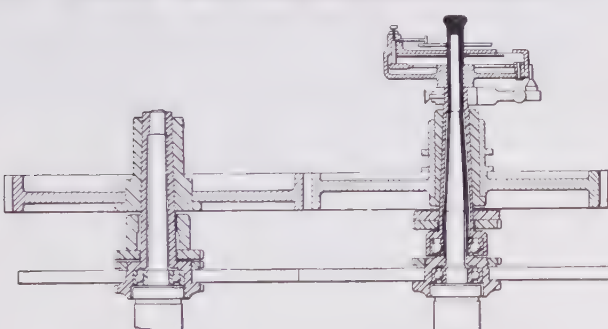


Fig. 143.—Design of the Saurer testing machine.

the chart is mounted on the fixed spindle. When the first spindle is revolved, the gear mounted on it drives the gear mounted on the double sleeve, while the pitch disk on the first spindle drives its mating pitch disk by friction. The indicator records any

difference in the angular position of the two sleeves on the second spindle. If the gears are perfect, no angular movement takes place between these two sleeves, so that the indicator pointer remains stationary, and the chart will be a true circle. When errors are present in the gears, the indicator pointer moves accordingly, and the chart will show a correspondingly irregular line. If the gears are correct and the diameters of the pitch disks are correct, and provided no slippage takes place between them, the chart will be a perfect circle. An error in the diameters of the pitch disks or slippage between them results in a spiral chart.

As a matter of fact, a slightly spiral chart is often of advantage, as the gears may then be revolved two or three times, and the succeeding charts may then be closely compared with each other.

In Fig. 144 are shown charts made by pairs of 15-tooth gears of different degrees of accuracy. The actual charts are about 3 in. in diameter. A variation of about $\frac{1}{16}$ in. from a smooth line on the chart is caused by a difference of 1 min. in the angular position of the two sleeves on the indicator spindle. All of this error may be in one gear or it may be partly in one and partly in the other. If both are identical, a variation of $\frac{1}{16}$ in. on the chart would indicate an error of 30 sec. of arc in each gear. By testing the gears against an accurate master gear, the errors can be definitely located and measured. An analysis of the chart enables errors in spacing, profile, and concentricity to be determined very closely. An eccentric gear makes an eccentric chart. If the ratio is other than 1:1, the eccentricity develops lobes, which indicate the particular gear of the pair that is at fault. Errors in spacing show up as steps, while errors in profile develop irregular patterns, depending upon the nature of these errors.

Another somewhat similar testing machine, developed by the Gear Grinding Machine Company of Detroit, Michigan is the one shown in Fig. 145. In effect, it tests the gear against a master rack instead of its mating gear. The gear to be tested is mounted on a vertical spindle, which has an arm with an indexing arrangement so that the spindle may be rotated in increments of 1 deg. A tooth of this gear engages with a cone-shaped disk on the lower horizontal shaft. A section of this cone-shaped disk represents the profile of the basic rack. This horizontal shaft can move endwise, and the cone-shaped disk is held in contact with the gear tooth by a spring. The position of

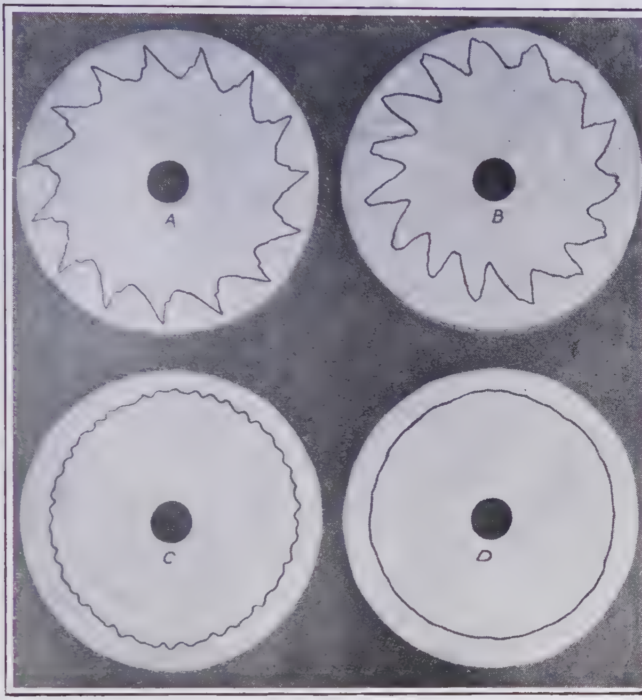


FIG. 144.—Typical charts produced on the Saurer testing machine with 15-tooth pinions.

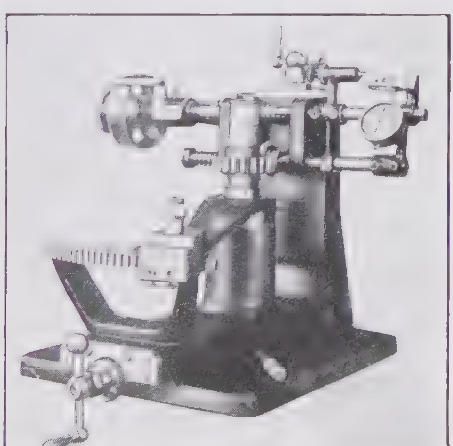


FIG. 145.—Rack-type gear-testing machine.

the upper horizontal shaft is controlled by a micrometer screw. A multiplying lever and dial indicator enable any difference in the endwise position of the two horizontal shafts to be measured.

In operation, the gear and the rack element are positioned at one end of the arc of contact, and the dial indicator is set to zero. The gear is then rotated 1 deg., and the upper shaft is moved by the micrometer screw a distance equal to the arc of 1 deg. on the pitch line of the gear. If the action between the gear and the basic-rack element is correct, the dial indicator will then be at zero again. Any deviation of the dial indicator represents a corresponding error in the action between the gear and its basic rack. This operation is repeated through the arc of action, and the deviations may be plotted on the same type of chart as shown in Fig. 137, or on standard cross-section paper.

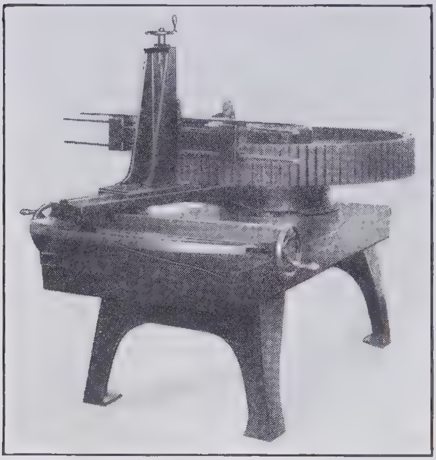


FIG. 146.—Similar type of machine for testing large gears.

Flat disks and needle-pointed or knife-edge elements are also used on this testing machine, thus making it universal in its action and enabling the exact profile of a gear to be determined as well as its smoothness of action with a basic-rack element. In these cases, the movement of the micrometer screw will be that corresponding to the length of the arc of rotation on the base circle. Other conjugate forms than the involute may also be tested on this machine by providing a basic-rack element of the proper form. Thus, the profile and action of gears made to the composite system may be tested on this machine by providing a disk with the profile of the basic composite rack of the proper pitch.

A larger instrument of this type is shown in Fig. 146. This testing machine will accommodate gears up to 72 in. in diameter. The principle of its operation is identical to that of the smaller instrument.

A modification of this type of testing machine is shown in Fig. 147. In this case, the gear is tested against an accurate master gear mounted on the same spindle. Both horizontal shafts are provided with cone-shaped disks representing the basic-rack profile. Any variation in the endwise positions of the two horizontal shafts in relation to each other as the gears are rotated is indicated on the dial indicator, as before. This instrument provides a very rapid means of testing gears in production, as a gear may be set up and tested in about 15 sec.

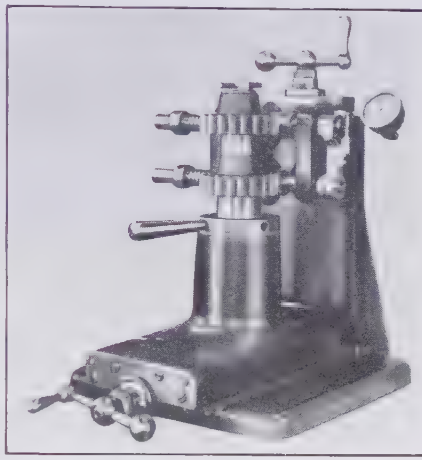


Fig. 147. Testing machine of basic-rack type employing a master gear.

These testing machines have the advantage of being positive in their action, as no pitch disks, tapes, or other friction devices are employed, thus eliminating error due to slippage.

Projection apparatus may also be used to make a composite test of the gear-tooth profiles. In this case, the image or the shadow of the meshing teeth of a pair of gears is projected on a screen, and the action between them may be studied as the gears are rolled together. If edge contact at the beginning of mesh occurs, it can be very readily discovered by this method of testing.

In order to test gears for quietness of operation, it is necessary to mount them and run them under load. This is sometimes done after the gears are assembled in position in the gear cases. At other times, a special fixture is provided, consisting of two spindles with means for adjusting the center distance. One spindle is driven, preferably by a belt drive, while the other one has a drum or brake by means of which a load can be applied. Very simple fixtures of this type are often made and attached to a lathe or plain milling machine. In this case, the spindle of the machine becomes the driving spindle, while the cross-slide or table provides means for adjusting the center distance.

CHAPTER X

HOBBING OF GEAR TEETH

The method of producing gears by form milling is limited to gears that run at relatively low pitch-line velocities because of the errors in form that inevitably develop from distortion of the form cutter in hardening, inaccuracies in the form of the cutter itself, and errors in setting up the machine. These conditions and also the search for methods of more rapid production led to the development of the hobbing process.

This process consists of revolving and advancing a worm-shaped cutter through a revolving blank. The ratio between the speed of the hob and that of the gear blank is determined by the number of threads or starts on the hob and the number of teeth to be cut in the gear blank. Thus, if a single-threaded hob is used to cut a 24-tooth gear, the hob would revolve 24 times while the gear blank revolves once. If a double-threaded hob were used, the hob would revolve 12 times while the gear blank revolved once.

In general, the hob is set at such an angle in relation to the teeth of the gear being cut that the helix at the middle of the tooth of the hob is tangent to the sides of the gear teeth. In Fig. 148, the cutter head and work arbor of a typical hobbing machine is shown.

Hobbing is a generating or molding process. A simple conception of this method is to consider any axial section of the hob as representing the basic rack. Successive axial sections, because of the lead or helix, in effect, will advance this basic rack in the direction of the axis of the hob as the hob is revolved. Thus, when the hob has completed a full revolution, the basic rack will have advanced one full-tooth interval. This process has the advantage of requiring but one hob to produce mating gears of any number of teeth. It also possesses the advantage over form milling of enabling more effective tooth proportions to be used when desired, using the same cutting tool, while a special form cutter would be required in each individual case with form milling. For

quantity production, in particular, hobbing has proven to be a rapid and effective method. All motions are continuous, and, except for the feeding, all motions are rotary. Such continuous rotary motions are usually conducive to rapid production.

The many advantages of this method of production soon led to its wide adoption. But it soon became apparent that this newer process in itself did not eliminate all of the difficulties of producing accurate gears. There are still many problems to be solved

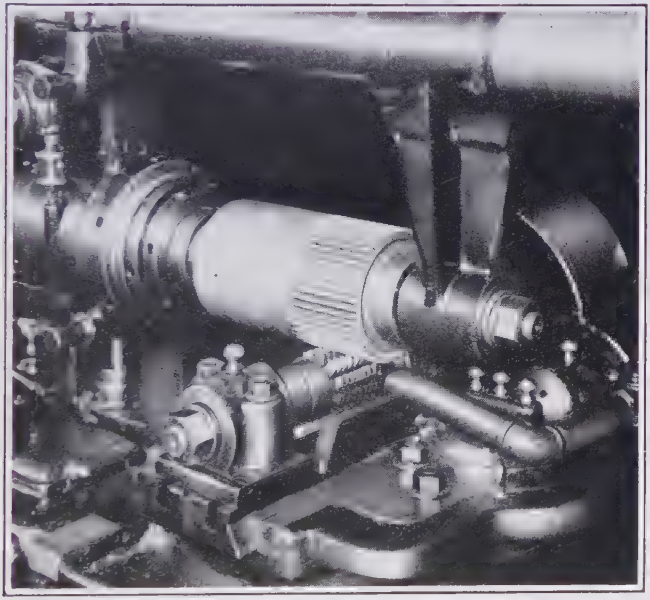


FIG. 148.—Gear being cut by a typical hob.

before that end is reached, some in relation to the machines and others in relation to the hobs. We will first direct our attention to the problems in relation to the hobbing machines.

Many types of hobbing machines are now on the market. All of them, however, follow certain general lines. The gear train between the hob and the work arbor, which carries the gear blanks, in its simplest form is shown in the diagram, Fig. 149. In addition to the movements shown, provision must be made to allow for the adjustment of the hob to suit both the diameter of the gear blank and the helix angle of the hob. Provision must also be made to obtain the necessary feeding of the hob through the

gear blank. These motions have been omitted from the diagram to simplify it.

The accuracy of the product depends upon three primary factors: first, the accuracy of the hobbing machine and work-holding fixture; second, the accuracy of the hob; and third, the carefulness of the operator in setting up the gear blanks and the care exercised in resharpening the hobs.

Inaccuracies in the functional parts of the hobbing machine will be reproduced in the product. The work spindle and arbor on which the blanks are mounted must be concentric, otherwise eccentric gears will be produced. The index worm and worm-

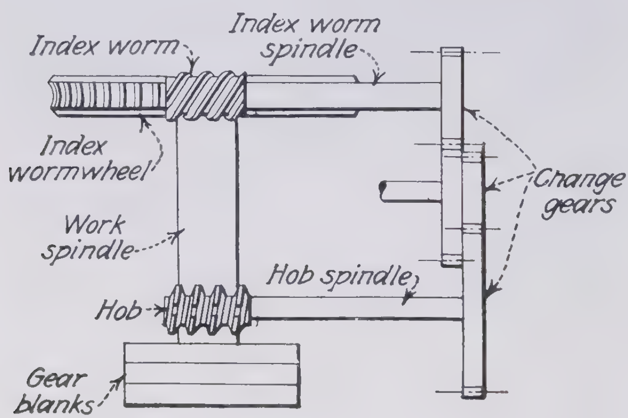


FIG. 149.—Simplified gear train connecting the hob and work arbors.

wheel are also vital elements of the hobbing machine. The index worm and spindle must be concentric and must run true, else an irregular motion will be imparted to the wormwheel and work spindle that will result in troublesome tooth-profile errors on the product. The index wormwheel, in turn, must be concentric with the work spindle. Any eccentricity in the wormwheel will cause the work spindle to accelerate for half a revolution and decelerate for the remaining half, thus introducing errors in the tooth profiles of the product that will be identical in their results to the same amount of eccentricity in the mounting of the gear blanks. The teeth in the index wormwheel must also be accurately spaced. Inaccuracies of spacing here will be reproduced in the product partially as spacing errors and partially as tooth-profile errors.

The cutter or hob spindle must also be concentric and run true. Any eccentricity or cam action of imperfect thrust bearings will be reproduced in the product in the form of troublesome tooth-profile errors.

The most important factor of the change gears is their concentricity. This is one application where eccentricity in gears is a serious defect. In such a train of gears as this, the cumulative error in position caused by eccentricity will be reproduced in the product both as spacing and tooth-profile errors. The effect of this eccentricity can be reduced by having the tooth numbers of all gears in the train an even multiple of the tooth number of the smallest gear in the train. Thus, if the smallest gear has 18 teeth, the tooth numbers of all the other gears should be some multiple of 18, as 54, for example.

Backlash in the thrust bearings of either the index-worm spindle or the hob spindle will result in faulty tooth profiles on the product. The cutting thrust between the hob and the gear blanks varies during the cutting cycle, both in direction and amount. Sometimes the cutting is all on one side of the hob tooth, then on both sides, and then on the opposite side only. In order to reproduce the maximum accuracy of the machine, two finishing cuts should be taken, neither of them cutting at the root of the tooth space of the gear blanks. The first finishing cut would be taken on one side of the gear tooth only, while the second cut would finish the opposite side of the gear tooth. In this way, the cutting thrust between the hob and the gear blanks is held in one direction only, which tends to minimize the effects of backlash in the thrust bearings of the index-worm and the hob spindles, and also in change gears and other connections between the hob spindle and the work spindle.

The construction of the hobbing machine must be sufficiently rigid to prevent excessive deflection, torsion, or vibration of the machine and its operating parts under the cutting load. The advantages of the accuracy built into the machine will be lost entirely if its construction is not sufficiently rigid. The rigidity of the machine controls, in large measure, the maximum rate of production. Slower cutting speeds and finer feeds must be used when the construction of the machine is not stiff.

Hobbing machines are made in many types. Some have horizontal work spindles while others have vertical work spindles. In general, the vertical work spindles are used on the larger machines,

as this construction offers better opportunities of clamping the work. The smaller gears are usually cut in multiple on arbors, so that the position of the work spindle is of secondary importance.

The larger hobbing machines and some of the smaller ones are made of a universal-purpose type. Other smaller machines are made of a single-purpose type, primarily for the rapid production of the gears used in large quantities for automotive construction. These machines are simplified as much as possible in their con-

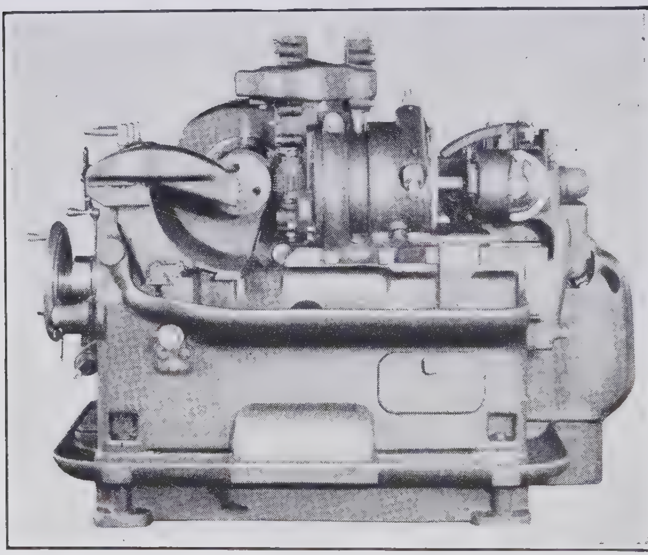


Fig. 150.—Hobbing machine of rigid construction for cutting automotive gears.

struction and made as compact and rigid as possible. In Fig. 150, one machine of this type is shown.

Hobs for Involute Gears.—Let us now direct our attention to the problem of the hob. At the present time, this problem seems to be one of great importance. The hob consists of a practically straight-sided, worm-shaped cutter, the axial section of which represents the projected form of the basic rack of the gear-tooth system. If the axis of the hob were set at a right angle with the axis of the gear blank, its axial section would represent the form of the basic rack. As the hob is usually set at an angle to the gear blank that corresponds to the helix angle of the hob at its pitch

line, the angle of the hob tooth and its lead must be altered accordingly.

Thus, let
α = angle of side of basic rack
β = angle at which hob is set
α'' = angle of side of hob tooth
P = circular pitch of basic rack
K = lead of hob

Then

$$\tan \alpha'' = \frac{\tan \alpha}{\cos \beta} \tag{127}$$

$$K = \frac{P}{\cos \beta} \text{ for single-threaded hobs} \tag{128}$$

$$K = \frac{2P}{\cos \beta} \text{ for double-threaded hobs} \tag{129}$$

$$K = \frac{nP}{\cos \beta} \text{ for multiple-threaded hobs, where } n = \text{number of threads or starts} \tag{130}$$

A common fault of gears produced with the ordinary form of hob is that the portion of the involute profile near the tip of the tooth does not make proper contact. Using shop terms, the gear teeth often have a "low bearing." In other words, the involute profile seems to be distorted in such a manner that additional metal is removed at this point. A slight modification at this point is often beneficial, but too much will shorten the duration of contact. On gears with relatively small tooth numbers, where the duration of contact is short at best, excessive modification at this point results in noisy gears.

Multiple-threaded hobs prove much faster producers than single-threaded ones, but they are seldom used except for roughing cuts because of the large errors that develop in the tooth profiles of the product. The cause of these errors is one problem of the hob to be considered.

As stated before, the hob is usually set at an angle to the gear blank. Some hobs are now made for spur gears that are set square with the axis of the gear blank. Considerable uncertainty exists in regard to the effect of setting a hob at various angles. The fundamental differences, if any, between a hob set at an angle and a hob set square is another problem to be studied. The effect of varying the angular setting of the hob is still another problem.

Many of the difficulties of hobbing accurate gears are blamed on the distortion of the hob in hardening. Such distortion is a serious handicap and will always be one of the most difficult problems to solve. Grinding the hob after hardening is possible but expensive, particularly grinding the relieved surfaces of a hob. Regardless of hardening defects, however, if it should prove that the ordinary practice in making hobs introduces defects in the form of a product that can be eliminated, the chances of getting a good hob through the hardening process would be greatly increased.

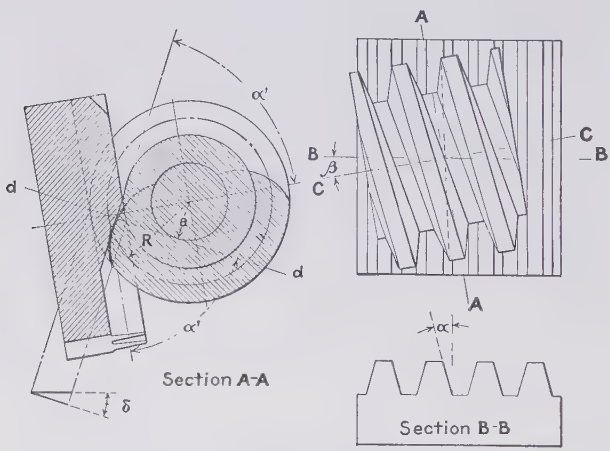


Fig. 151.—Theoretical rack and hob.

An involute gear is generated from a straight-sided rack. Theoretically, a hob that will generate a true involute gear will also generate a straight-sided rack. Although, for practical reasons, a hob would never be used to cut a rack, we will use one in the study of the hob, because its straight-line profile is the easiest one to deal with.

The simplest example of such a hob is an involute helical gear of one tooth (a single-threaded worm, in other words) running with a straight-sided rack. For the present, all considerations of relief, nature of cutting flutes, and other refinements will be ignored. This hob consists of an infinite number of spur gears twisted uniformly to give the lead of the helix. In Fig. 151, such a hob and rack are illustrated.

In this illustration, let

a = radius of base circle of involute in section AA
R = radius of pitch circle of involute
α = angle of side of straight-sided rack tooth
α' = pressure angle of involute in section AA, which is also the projected angle of α in section AA
β = angle at which hob is set, which is also the helix angle of the hob at radius R
M = module of rack $= \dfrac{1}{\text{diametral pitch}}$
P = circular pitch of rack $= \pi M$
K = lead of hob

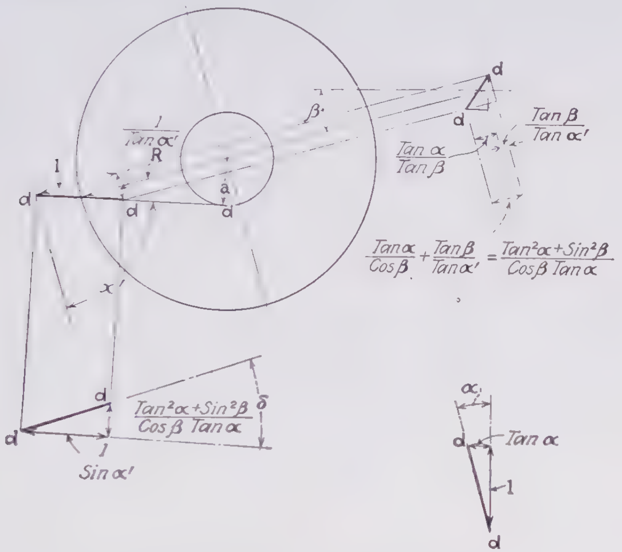


Fig. 152.—Diagrammatic layout of contact between rock and hob.

Referring to Fig. 152, we have

$$\tan a = \frac{\tan \alpha}{\sin \beta} \tag{131}$$

$$K = \frac{P}{\cos \beta} \qquad \text{(see Eq. (128))}$$

$$\tan \beta = \frac{K}{2\pi R} = \frac{P}{2\pi R \cos \beta} = \frac{M}{2R \cos \beta} \tag{132}$$

whence,

$$\sin \beta = \frac{M}{2R} \text{ or } R = \frac{M}{2 \sin \beta} \tag{133}$$

In section AA, the line dd is perpendicular to the side of the projected section of the rack tooth and passes through the intersection of the center line and pitch circle of the hob. This line dd is therefore the line of action of the involute and the rack. We already know that contact between an involute and a straight line can take place only along such a line of action. The circle concentric with the hob to which this line dd is tangent will be the base circle of the involute. Whence,

$$a = R \cos \alpha' \tag{134}$$

Considering now the infinite number of successive sections of the involute which are twisted uniformly to the lead of the hob, we see that contact between the straight-sided rack and hob must be in a straight line, because the projection of the line of contact in the plane AA is the straight line dd, and this line of contact also lies in the plane of the straight-sided rack tooth. If the angle of this line of contact with a plane perpendicular to the axis of the hob is determined, we would generate the surface of the teeth of a hob of correct form by revolving this line of contact, advancing it according to the lead of the hob and keeping it tangent to a cylinder of the same diameter as the base circle in section AA, just determined. In section AA, this line is at the angle α' from the horizontal. In section BB, it must be at the angle α from the vertical, because this line of contact must be in the plane of the side of the rack. The angle δ in section dd is the desired angle. Figure 152 shows only the lines required to establish this angle. It is merely a problem of descriptive geometry. The contact line dd is shown in heavy lines in the four projections. The projection lines enable their construction to be followed. The values of the lengths of the different lines are indicated on the diagram. From these, we have

$$\tan \delta = \frac{(\tan^2 \alpha + \sin^2 \beta) \sin \alpha'}{\cos \beta \tan \alpha}$$

But

$$\sin \alpha' = \frac{a \tan \alpha}{R \sin \beta}$$

Substituting in the foregoing equation, we have

$$\tan \delta = \frac{a(\tan^2 \alpha + \sin^2 \beta)}{R \sin \beta \cos \beta}$$

But

$$\tan^2 \alpha = \sin^2 \beta \tan^2 \alpha'$$

Substituting, we have

$$\tan \delta = \frac{a \sin \beta(\tan^2 \alpha' + 1)}{R \cos \beta} = \frac{a \tan \beta(\tan^2 \alpha' + 1)}{R}$$

$$\tan \beta = \frac{K}{2\pi R}$$

Substituting, we have

$$\tan \delta = \frac{aK(\tan^2 \alpha' + 1)}{2\pi R^2}$$

$$(\tan^2 \alpha' + 1) = \frac{1}{\cos^2 \alpha'}$$

Substituting, we have

$$\tan \delta = \frac{aK}{2\pi R^2 \cos^2 \alpha'}$$

But

$$R \cos \alpha' = a$$

whence,

$$\tan \delta = \frac{K}{2\pi a} \tag{135}$$

But the tangent of the helix angle of the hob at radius a is equal to $\frac{K}{2\pi a}$

The surface of a hob, therefore, that will generate a straight-sided rack or an involute gear is developed by revolving and advancing with uniform lead a straight line inclined at any given angle to the plane perpendicular to the axis of the hob, provided this line remains tangent to a cylinder of such a diameter that the helix angle at that diameter is the same as the angle between the generatrix and a plane perpendicular to the axis.

The characteristics of such a surface can be visualized by winding a string around a cylinder on a helix, as shown in Fig. 153. The cylinder would represent the base cylinder of this involute helicoidal surface. Both ends of the string can be tied around the cylinder, one at each end. The bight of the string should then be held so that at one end of the cylinder the string is perpendicular to the axis and at the other end it is at an angle. Then, when the cylinder is revolved, winding the string about it, one part of the string will be wrapped around in the form of a helix while the other part will be wound upon itself. The angle at the bight of the string will remain constant, and that part of the string which is at an angle will be a continua-

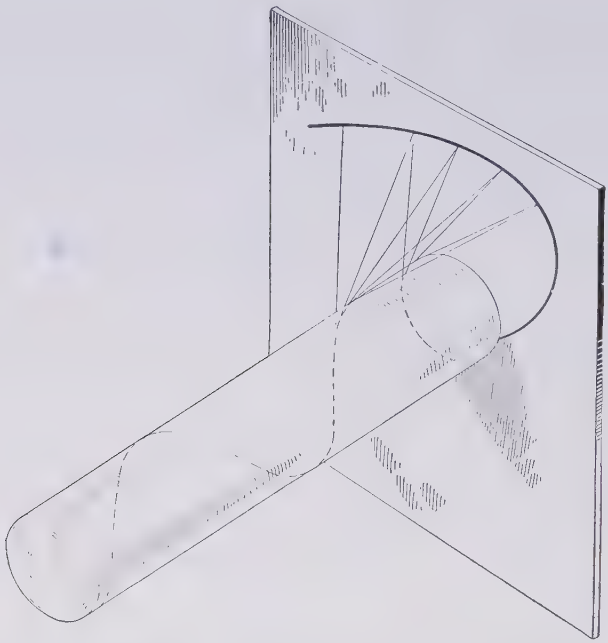


Fig. 153.—Generation of the involute-helicoidal surface of hob.

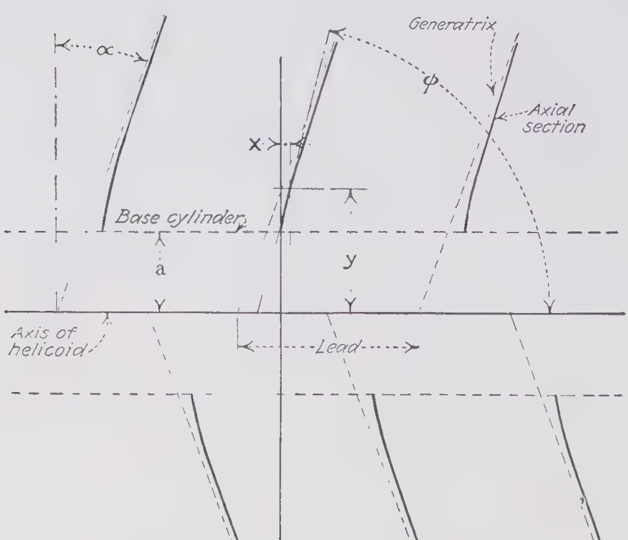


Fig. 154.—Axial sections of an involute helicoid, general case.

tion of the helix angle on the base cylinder. As the string is unwound again, the corner of the bight will trace an involute curve on a plane perpendicular to the axis of the cylinder, while the angular part of the string would develop an involute helicoidal surface. It will be noted that the straight-line form of this helicoidal surface will be its intersection curve with a plane tangent to its base cylinder and not its intersection curve with an axial plane. This feature of the involute helicoid makes it simple to determine the characteristics not only of hobs but also of helical involute gears whose surfaces are also involute helicoids.

The equation of the axial section of an involute helicoid, shown in Fig. 154, is as follows:

$$x = \frac{K}{2\pi}\left(\frac{\sqrt{y^2 - a^2}}{a} - \arctan\frac{\sqrt{y^2 - a^2}}{a}\right) \tag{136}$$

Where x = distance along the axis of the helicoid
y = radius to any point of the helicoid

The equation of the tangent to this curve, measured from the X-axis, is as follows:

$$\tan\phi = \frac{dy}{dx} = \frac{2\pi a y}{K\sqrt{y^2 - a^2}} \tag{137}$$

When the tangent at the pitch line is desired, as is usually the case, y becomes equal to R, and the equation of the tangent reduces to the following:

$$\tan\phi = \frac{2\pi a R}{K\sqrt{R^2 - a^2}} \tag{138}$$

$$\frac{2\pi R}{K} = \frac{1}{\tan\beta}$$

$$a = \frac{R\sin\beta}{\sqrt{\tan^2\alpha + \sin^2\beta}}$$

Substituting these values, we obtain

$$\tan\phi = \frac{R\sin\beta}{R\tan\beta\tan\alpha} = \frac{\cos\beta}{\tan\alpha} \tag{139}$$

From the preliminary analysis, Eq. (127), we have

$$\tan\alpha'' = \frac{\tan\alpha}{\cos\beta}$$

whence,

$$\tan\phi = \frac{1}{\tan\alpha''}$$

As definite examples, we will calculate and plot the profiles of single-, double-, and triple-threaded hobs of the same diameter and pitch. We will take 6-d.p. hobs, 14½-deg. pressure angle, with a pitch diameter of 3 in. For the single-threaded hob, we obtain the following values:

$$\alpha = 14\tfrac{1}{2} \text{ deg.}$$
$$R = 1.500 \text{ in.}$$
$$M = \tfrac{1}{6}$$
$$\sin \beta = \frac{M}{2R} = \frac{1}{6 \times 3} = 0.05556$$
$$\beta = 3 \text{ deg. } 11 \text{ min. } 5 \text{ sec.}$$
$$K = \frac{P}{\cos \beta} = \frac{0.5236}{0.99845} = 0.52441 \text{ in.}$$
$$a = \frac{R \sin \beta}{\sqrt{\tan^2 \alpha + \sin^2 \beta}} = \frac{0.08333}{0.31264} = 0.31503 \text{ in.}$$

Solving the equation for the profile, we get

y	x	$(x - 0.27405)$
1.3333	0.23116	−0.04289
1.3667	0.23972	−0.03433
1.4000	0.24828	−0.02577
1.4333	0.25686	−0.01719
1.4667	0.26545	−0.00860
1.5000	0.27405	0
1.5333	0.28265	+0.00860
1.5667	0.29126	+0.01721
1.6000	0.29989	+0.02584
1.6333	0.30852	+0.03447
1.6667	0.31715	+0.04310

Solving the equation for the tangent at the pitch line, we get

$$\tan \phi = 3.86068, \quad \phi = 75 \text{ deg. } 28 \text{ min. } 45 \text{ sec.}$$

whence,

$$\alpha = 14 \text{ deg. } 31 \text{ min. } 15 \text{ sec.}$$

These values are plotted in Fig. 155. The theoretical profile of the axial section of this hob is practically a straight line, departing from it about 0.0003 in. at the root of the hob tooth and less

than 0.0001 in. at the tip of the hob tooth. Thus, if this hob were made with a straight-line axial section, it would introduce a slight modification at the tip of the hobbed gear of less than 0.0003 in. in the direction that tends to avoid edge contact.

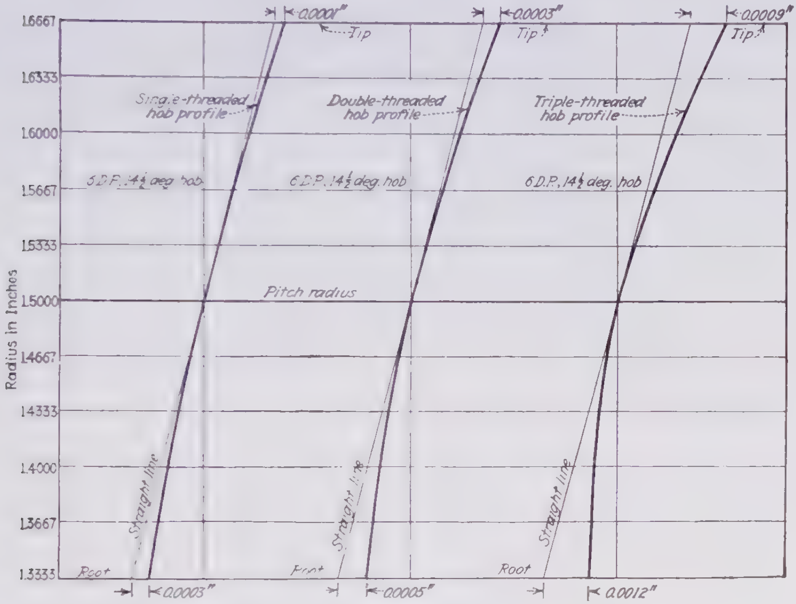


FIG. 155. FIG. 156. FIG. 157.

FIG. 155.—Axial section of a single threaded-hob.
FIG. 156.—Of a double-threaded hob.
FIG. 157.—Of a triple-threaded hob.

We will now direct our attention to the double-threaded hob of the same pitch and diameter. For this, we have the following values

$$\alpha = 14\tfrac{1}{2} \text{ deg.}$$
$$R = 1.500 \text{ in.}$$
$$M = \tfrac{1}{6}$$
$$\sin\beta = \frac{2M}{2R} = \frac{2}{3 \times 6} = 0.11111$$
$$\beta = 6 \text{ deg. } 22 \text{ min. } 46 \text{ sec.}$$
$$K = \frac{2P}{\cos\beta} = \frac{2 \times 0.5236}{0.99381} = 1.05372 \text{ in.}$$
$$a = \frac{R\sin\beta}{\sqrt{\tan^2\alpha + \sin^2\beta}} = \frac{0.16667}{0.28147} = 0.59213 \text{ in.}$$

Solving the equation for the profile, we get

y	x	$(x - 0.19496)$
1.3333	0.15209	−0.04287
1.3667	0.16058	−0.03438
1.4000	0.16910	−0.02586
1.4333	0.17768	−0.01728
1.4667	0.18630	−0.00866
1.5000	0.19496	0
1.5333	0.20365	+0.00869
1.5667	0.21237	+0.01741
1.6000	0.22113	+0.02617
1.6333	0.22991	+0.03495
1.6667	0.23872	+0.04376

Solving the equation for the tangent at the pitch line, we get

$$\tan \phi = 3.84274, \quad \phi = 75 \text{ deg. } 24 \text{ min. } 48 \text{ sec.}$$

whence,

$$\alpha'' = 14 \text{ deg. } 35 \text{ min. } 12 \text{ sec.}$$

These values are plotted in Fig. 156. This profile is also practically a straight line, but it shows a greater departure than on the single-threaded hob. In this case, the departure from a straight line at the root of the hob tooth is about 0.0005 in., while it is about 0.0003 in. at the tip. These amounts are roughly double those on a single-threaded hob. This departure introduces a greater modification in the tooth profiles of the hobbed gear when a straight-sided axial section hob is used than before.

We will now direct our attention to the triple-threaded hob of the same pitch and diameter. For this, we have the following values:

$$\alpha = 14\tfrac{1}{2} \text{ deg.}$$
$$R = 1.500 \text{ in.}$$
$$M = \tfrac{1}{6}$$
$$\sin \beta = \frac{3M}{2R} = \frac{3}{3 \times 6} = 0.16667$$
$$\beta = 9 \text{ deg. } 35 \text{ min. } 40 \text{ sec.}$$
$$K = \frac{3P}{\cos \beta} = \frac{3 \times 0.5236}{0.98601} = 1.59309 \text{ in.}$$
$$a = \frac{R \sin \beta}{\sqrt{\tan^2 \alpha + \sin^2 \beta}} = \frac{0.25000}{0.30767} = 0.81256 \text{ in.}$$

Solving the equation for the profile, we get

y	x	$(x - 0.14031)$
1.3333	0.09775	−0.04256
1.3667	0.10605	−0.03426
1.4000	0.11447	−0.02584
1.4333	0.12299	−0.01732
1.4667	0.13162	−0.00869
1.5000	0.14031	0
1.5333	0.14909	+0.00878
1.5667	0.15794	+0.01763
1.6000	0.16687	+0.02656
1.6333	0.17586	+0.03555
1.6667	0.18492	+0.04461

Solving the equation for the tangent at the pitch line, we get

$$\tan \phi = 3.81258, \ \phi = 75 \text{ deg. } 18 \text{ sec. } 11 \text{ min.}$$

whence

$$\alpha'' = 14 \text{ deg. } 41 \text{ min. } 49 \text{ sec.}$$

These values are plotted in Fig. 157. The profile here makes a noticeable departure from a straight line, departing about 0.0012 in. at the root of the hob tooth and about 0.0009 in. at the tip. In this case these amounts are over four times as great as those on a single-threaded hob. This would introduce too much modification on the tooth profiles of a gear produced with a straight-sided hob, if quietly running gears are required.

With the same diameter of hob, as the pitch becomes greater, the departure from a straight-line profile will also become greater, because both the helix angle and the depth of tooth will increase.

It should be apparent, from the foregoing, why single-threaded hobs with straight-line tooth profiles prove more satisfactory than multiple-threaded ones. There is no theoretical reason, however, that would prevent more satisfactory multiple-threaded hobs from being made if the profile is properly corrected. This can be accomplished in a simple manner by setting the straight-sided tool used to cut the threads on the hob a suitable distance off center.

Influence of the Setting Angles of the Hob.—The angle β at which a hob is set may be chosen, within practical limits, at random, and a hob can be developed to cut a straight-sided rack or involute gear of the desired pitch. The angle β may

be reduced to zero, which would result in the conditions shown in Fig. 158. Referring to previous equations, we have

$$\tan \alpha' = \frac{\tan \alpha}{\sin \beta} \tag{131}$$

where α' = pressure angle of hob in plane perpendicular to axis
and α = pressure angle of rack tooth

When

$$\beta = 0$$
$$\sin \beta = 0$$
$$\tan \alpha' = \frac{\tan \alpha}{0} = \infty$$

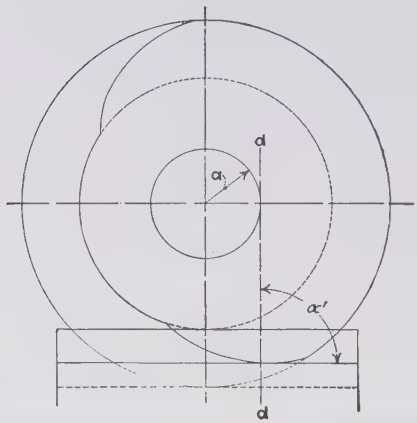


Fig. 158.—Diagram of hob set square with rack.

whence,

$$\alpha' = 90 \text{ deg.}$$

Let R = pitch radius of hob
M = module of rack
δ = angle of generatrix of theoretical hob

$$R = \frac{M}{2 \sin \beta} = \frac{M}{0} = \infty$$

$$\tan \delta = \frac{(\tan^2 \alpha + \sin^2 \beta) \sin \alpha'}{\cos \beta \tan \alpha}$$

$\beta = 0$ whence, $\sin \beta = 0$, and $\cos \beta = 1$
$\alpha' = 90$ whence, $\sin \alpha' = 1$

$$\tan \delta = \frac{(\tan^2 \alpha + 0) \times 1}{1 \times \tan \alpha}$$
$$= \tan a \text{ whence, } \delta = \alpha$$

Thus, when the angle β at which the hob is set is equal to zero, the angle δ of the generating line is equal to the angle of the rack. The pressure angle of the involute curve in section dd in Fig. 158 will be 90 deg., and the radius of the pitch circle will be infinity. With screw gears, or helical involute gears whose axes are not parallel, the pitch circles or cylinders of the mating gears are not necessarily tangent to each other. The case just cited is one example of such a condition.

When the hob is set with a helix tangent to the tooth of the rack or gear at any given diameter of the hob, the normal thickness of the hob tooth at that point is equal to the width of the space at the mating portion of the rack or gear. In all other cases, the space cut by the hob tooth will be greater than the normal thickness of the hob tooth. Therefore, if a predetermined width or space must be maintained, the thickness of the hob tooth must be reduced accordingly when the hob is set to some other angle than that of its helix at some predetermined diameter. This reduction in the thickness of the hob tooth is one limitation in regard to the selection of the angle at which it is to be set. A more serious limitation in this respect, however, is the height of the resulting fillet at the root of the hobbed rack or gear.

Before proceeding further, it is of interest to determine the effect on the form of an involute gear tooth of a variation in the setting angle β. Referring to previous equations, we have the following:

$$K = \frac{P}{\cos \beta} = \text{lead of hob} \qquad \text{(see Eq. (128))}$$

whence,

$$P = K \cos \beta = \text{circular pitch of basic rack}$$

$$\tan \delta = \frac{K}{2\pi\alpha} \qquad \text{(see Eq. (135))}$$

$$\alpha = R \cos \alpha' = \text{radius of base cylinder of hob} \qquad \text{(see Eq. (134))}$$

$$\tan \alpha' = \frac{\tan \alpha}{\sin \beta}, \qquad \text{(see Eq. (131))}$$

and

$$M = \frac{P}{\pi}$$

$$R = \frac{M}{2 \sin \beta} = \frac{P}{2\pi \sin \beta} \qquad \text{(see Eq. (133))}$$

whence,

$$\frac{\sin \alpha'}{\cos \alpha'} = \frac{\tan \alpha}{\sin \beta}$$

But

$$\frac{1 - \cos^2 \alpha'}{\cos^2 \alpha'} \quad \frac{\tan^2 \alpha}{\sin^2 \beta}$$

whence,

$$\cos \alpha' = \frac{\sin \alpha}{\sqrt{\tan^2 \alpha + \sin^2 \beta}}$$

$$a = R \cos \alpha' = \frac{P}{2\pi\sqrt{\tan^2 \alpha + \sin^2 \beta}}$$

$$\tan \delta = \frac{K}{2\pi a} = \frac{K\sqrt{\tan^2 \alpha + \sin^2 \beta}}{P}$$

$$= \frac{\sqrt{\tan^2 \alpha + \sin^2 \beta}}{\cos \beta}$$

whence,

$$\cos \delta = \cos \alpha \cos \beta \tag{140}$$

It will be seen, from the foregoing equation, that δ is dependent only on the angles α and β.

With an involute spur gear, if the pitch diameter is changed while the involute profile remains unchanged, the module and pressure angle change accordingly but maintain the following relationship:

Let M_1 = original module of gear
M_2 = changed module of gear
α_1 = original pressure angle
α_2 = changed pressure angle

then

$$M_1 \cos \alpha_1 = M_2 \cos \alpha_2$$

whence

$$M_1 = \frac{M_2 \cos \alpha_2}{\cos \alpha_1}$$

If this change in the module of the gear is caused by a change in the angular setting of the hob β, the new module will have the following relationship with the original one:

Let β_1 = original setting of hob
β_2 = changed setting of hob

$$K = \frac{P}{\cos \beta} = \frac{\pi M_1}{\cos \beta_1} = \frac{\pi M_2}{\cos \beta_2}$$

whence,

$$M_1 = \frac{M_2 \cos \beta_1}{\cos \beta_2} = \frac{M_2 \cos \alpha_2}{\cos \alpha_1}$$

$$\cos \alpha_1 \cos \beta_1 = \cos \alpha_2 \cos \beta_2$$

Referring to a previous equation, we have

$$\cos \delta = \cos \alpha \cos \beta = \cos \alpha_1 \cos \beta_1 = \cos \alpha_2 \cos \beta_2$$

Regardless, therefore, of the angle at which a hob is set, it produces the same involute on the gear being cut. The thickness of the space and tooth will change as the setting angle is altered, and the height of the fillet at the root of the tooth will change, but these will be the only differences.

It is evident that this freedom in choosing the angle at which to set the hob is of great practical value. Machine settings are never mathematically exact and do not need to be for hobs. The hob may be set to depth and angle by trial to secure the desired tooth thickness, and no theoretical error will develop. Furthermore, the thickness of the hob tooth itself is not important, so long as it is not too thick. A slight change in its angular setting on the machine will automatically compensate for any errors in tooth thickness.

A further advantage of this freedom in choosing the setting angle of the hob is the choice it gives of an exact lead for the hob instead of an approximate one. For instance, in a preceding example of a 6-d.p. hob of 3-in. pitch diameter, 14½-deg. pressure angle, the theoretical lead required was 0.52441 in. If the even fraction $^{11}/_{21}$ is used for the lead, this gives an approximation equal to 0.52381 in. A slight change in angle will compensate for any small difference between the theoretical lead and that actually used. All that is required is to have the correct normal pitch on the involute helicoid containing the cutting edges of the hob; this normal pitch is the shortest distance from one tooth surface to the next.

The normal pitch of a single-threaded hob is equal to $K \cos \delta$. This is also equal to $K \cos \alpha \cos \beta$, or $P \cos \alpha$. Thus, when

K = theoretical lead of hob
R = pitch radius of hob
α = theoretical normal pressure angle
β = angular setting of theoretical hob
δ = angle of generatrix of theoretical hob
K_1 = actual lead of hob
α_1 = corrected normal pressure angle
β_1 = corrected angular setting of hob, or helix angle at the pitch radius
δ_1 = corrected angle of generatrix of hob

$$K \cos \delta = K_1 \cos \delta_1$$

$$\cos \delta_1 = \frac{K \cos \delta}{K_1} \tag{141}$$

$$\tan \beta_1 = \frac{K_1}{2\pi R} \tag{142}$$

$$\cos \alpha_1 = \frac{\cos \delta_1}{\cos \beta_1} \tag{143}$$

In the example given,

$$K = 0.52441$$
$$R = 1.5000 \text{ in.}$$
$$\alpha = 14\tfrac{1}{2} \text{ deg.}$$
$$\beta = 3 \text{ deg. } 11 \text{ min. } 5 \text{ sec.}$$
$$\cos \delta = \cos \alpha \cos \beta = 0.96815 \times 0.99845 = 0.96665$$
$$K_1 = 0.52381 \text{ in.}$$

whence,

$$\cos \delta_1 = \frac{0.52441 \times 0.96665}{0.52381} = 0.96776$$

$$\tan \beta_1 = \frac{0.52381}{3 \times 3.1416} = 0.05558$$

$$\beta_1 = 3 \text{ deg. } 10 \text{ min. } 52 \text{ sec}$$

$$\cos \alpha_1 = \frac{0.96776}{0.99846} = 0.96925$$

$$\alpha_1 = 14 \text{ deg. } 14 \text{ min. } 40 \text{ sec.}$$

When the hob is set at such an angle that the helix at any diameter of the hob is tangent to the teeth of the gear being cut, the width of the space of the gear at that point will be equal to the normal thickness of the hob tooth at that diameter. At all other diameters of the hob, the space cut in the gear blank will be wider than the normal thickness of the hob tooth.

As noted before, a change in the angular setting of a hob affects only the width of space and the fillet at the root of the gear tooth. Above the fillet, the involute profile will remain unchanged. If a hob were used to cut a rack, a change in its angular setting would also affect its pressure angle and circular pitch, but the change in one element would compensate for the change in the other, so that both racks would have the same normal pitch and would mesh properly with the same involute gears.

The change in thickness of space is of secondary importance. The critical factor is the height of the fillet. Too high a fillet at the root of a gear tooth leads to edge contact at the beginning

of mesh, a most serious fault. In many cases, interference on the fillet is responsible for noisy and troublesome gears. This height of fillet limits the amount of variation in the angular setting of a hob that can safely be used. We will therefore direct out attention to the fillets that result from various angular settings of a hob.

For the purpose of simplicity, we will first consider the hob as generating a straight-sided rack. The fillet on a gear tooth will be greater in height than that on a rack, the amount depending largely upon the number of teeth in the gear. The smaller the number of teeth, the higher this fillet will be.

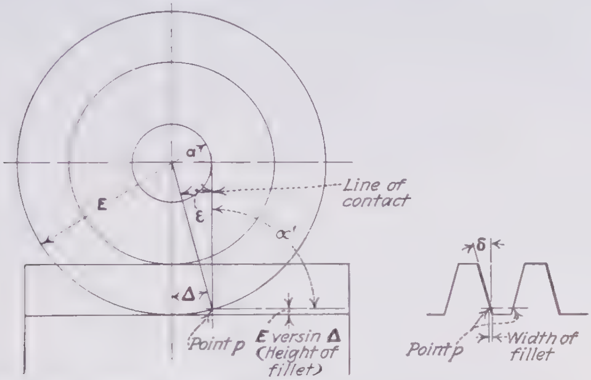


FIG. 159.—Determination of fillet formed by hob set square with rack.

We will also first consider a hob with sharp corners at the tips of the hob teeth. The actual fillets produced would be larger, both in width and height, by an amount approximately equal to the radius of the fillet at the tip of the hob tooth.

We will consider first a hob set square with the face of a rack. Referring to Fig. 159 let

K = lead of hob
a = radius of base cylinder
E = radius to tip of hob tooth
δ = angle of generatrix of hob
Δ = angle of rotation of hob from vertical center line to point of contact of point p
ϵ = angle between radial line to point p and line of contact

In this example,

$$\epsilon = \Delta \text{ and } \sin \epsilon = \frac{a}{E}$$

The additional height of the fillet is equal to the distance from the root of the rack to point p. Whence,

$$\text{Additional height of fillet} = E \text{ versin } \Delta \tag{144}$$

The additional width of the fillet is equal to E versin Δ tan δ plus the amount of the point p travels along its helix as it is revolved from the vertical center line to the line of contact. This point revolves through the angle Δ, whence,

$$\text{Additional width of fillet} = E \text{ versin } \Delta \tan \delta + \frac{\Delta K}{2\pi} \tag{145}$$

As a definite example, we will determine the additional height and width of the fillet produced on a rack by the hob used in the preceding example when set square with the face of the rack. This gives us the following values:

$$K = 0.52381 = {}^{11}\!/_{21} \text{ in.}$$
$$E = 1.6667 \text{ in.}$$
$$\cos \delta = 0.96776$$
$$\delta = 14 \text{ deg. } 35 \text{ min. } 20 \text{ sec.}$$
$$a = \frac{K}{2\pi \tan \delta} = \frac{11}{42 \times 3.1416 \times 0.26027} = 0.32031 \text{ in.}$$
$$\sin \delta = \frac{a}{E}\frac{0.32031}{1.6667} = 0.19218$$
$$\epsilon = \Delta = 11 \text{ deg. } 4 \text{ min. } 50 \text{ sec.}$$

$$\text{Additional height of fillet} = E \text{ versin } \Delta$$
$$= 1.6667 \times 0.01865 = 0.0311 \text{ in.}$$
$$\text{Additional width of fillet} = E \text{ versin } \Delta \tan \delta + \frac{\Delta K}{2\pi}$$
$$= 1.6667 \times 0.01865 \times 0.26027 + \frac{0.19339 \times 11}{42 \times 3.1416} = 0.0242 \text{ in.}$$

If the hob had a fillet of 0.0333 in. at the tip of its tooth, the total fillet at the root of the hobbed rack would be as follows:

$$\text{Height of fillet} = 0.0333 + 0.0311 = 0.0644 \text{ in.}$$
$$\text{Width of fillet} = 0.0333 + 0.0242 = 0.0575 \text{ in.}$$

These values are slightly greater than the actual ones, because the fillet at the corner of the hob tooth covers somewhat less than 90 deg. Any further refinement is not necessary, however.

In this case, the hob will generate a rack with a pressure angle of 13 deg. 35 min. 20 sec. The circular pitch of this rack will be the same as the lead of the hob, or equal to 0.52381 in. In Fig. 160, this rack form is plotted.

We will now consider the hob when it is set at an angle to the rack, the angle being in the same direction as the helix angle of the hob. This would give us the conditions shown in Fig. 161. In this case, let K, a, E, δ, Δ, and ϵ remain as before, and, in addition, let

α = pressure angle of hobbed rack
α' = angle of intersection of rack with plane perpendicular to axis of hob
P = circular pitch of hobbed rack
β = angle at which hob is set

Referring to the figure, we have

$$\sin \epsilon = \frac{E}{a}, \text{ and } \tan \alpha' = \frac{\tan \alpha}{\sin \beta}$$

$$\Delta = 90 \text{ deg} - (\alpha' + \epsilon)$$

$$\text{Additional height of fillet} = E \operatorname{versin} \Delta \qquad (146)$$

$$\text{Additional width of fillet} = E\,(\operatorname{versin} \Delta \tan a + \sin \Delta \sin \beta) - \frac{\Delta K}{2\pi \cos \beta} \qquad (147)$$

As a definite example, we will take the same hob as before and set it at an angle of 10 deg. in the same direction as the helix on the hob. This gives us the following values:

$$K = 0.52381 = 11/21 \text{ in.}$$

$$E = 1.6667 \text{ in.}$$

$$\epsilon = 14 \text{ deg. } 35 \text{ min. } 20 \text{ sec.}$$

$$\beta = 10 \text{ deg.}$$

$$a = 0.32031 \text{ in.}$$

$$\cos \alpha = \frac{\cos \alpha}{\cos \beta} = \frac{0.96776}{0.98481} = 0.98269$$

$$\alpha = 10 \text{ deg. } 40 \text{ min. } 30 \text{ sec.}$$

$$P = K \cos \beta = \frac{11 \times 0.98481}{21} = 0.51585 \text{ in.}$$

$$\sin \epsilon = \frac{a}{E} = \frac{0.32031}{1.6667} = 0.19218 \text{ in.}$$

$$\epsilon = 11 \text{ deg. } 4 \text{ min. } 50 \text{ sec.}$$

$$\tan \alpha' = \frac{\tan \alpha}{\sin \beta} = \frac{0.18850}{0.17365} = 1.08552$$

$$\alpha' = 47 \text{ deg. } 20 \text{ min. } 54 \text{ sec.}$$

$$\Delta = 90 \text{ deg.} - (\alpha' + \epsilon) = 90 \text{ deg.} - (11 \text{ deg. } 4 \text{ min. } 50 \text{ sec.} + 47 \text{ deg. } 20 \text{ min. } 45 \text{ sec.})$$

$$= 31 \text{ deg. } 34 \text{ min. } 16 \text{ sec.}$$

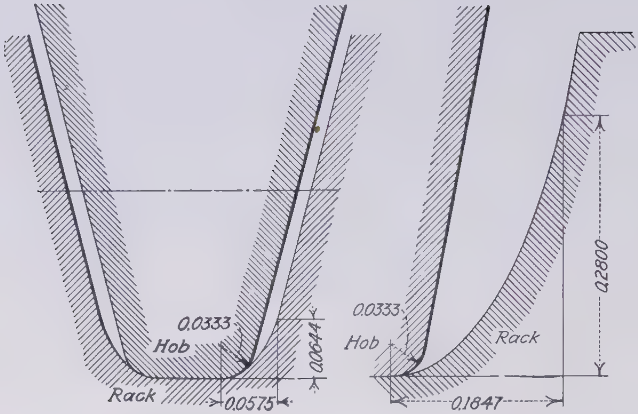


FIG. 160. FIG. 162.

FIG. 160.—Rack form developed by hob set square with it.
FIG. 162.—Rack form developed by hob set at an angle of 10 deg. to rack

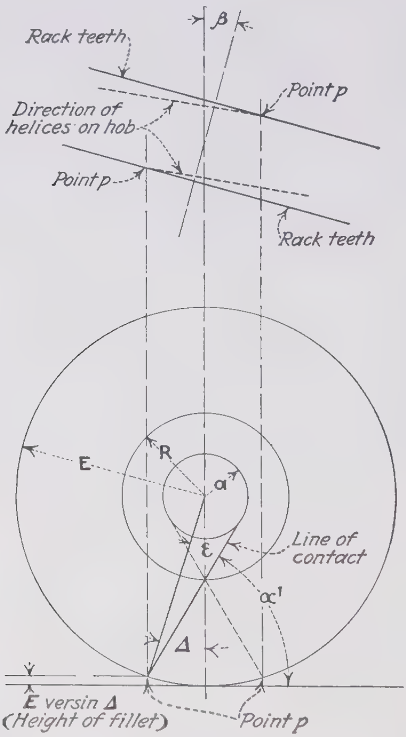


FIG. 161.—Diagram of hob set at an angle to rack.

Additional height of fillet

$$= E \text{ versin } \Delta = 1.6667 \times 0.14801 = 0.2467 \text{ in.}$$

Additional width of fillet

$$= E \text{ (version } \Delta \tan a + \sin \Delta \sin \beta) - \frac{\Delta K}{2\pi \cos \beta} =$$

$$1.6667(0.14801 \times 0.18850 + 0.52355 \times 0.17365) - \frac{0.55102 \times 11}{42 \times \pi \times 0.98481} = 0.1514 \text{ in.}$$

If the hob had a fillet of 0.0333 at the tip, the fillet at the root of the hobbed rack would be as follows:

Height of fillet $= 0.0333 + 0.2467 = 0.2800$ in.

Width of fillet $= 0.0333 + 0.1514 = 0.1847$ in.

In Fig. 162, this rack form is plotted.

This setting angle is so large that over half of the tooth profile of the hobbed rack would be in the fillet. In addition, the tooth would be thinner by an amount equal to about double the additional width of fillet, or about 0.3000 in. In this case, unless the tooth of the hob was very thin, there would be nothing but fillet for the rack tooth form. This setting angle is beyond the range of practical utility. If the hob were set at the helix angle at the outside diameter of the hob, there would be no side cutting at that diameter of the hob, and the fillet at the tip of the hob tooth would reproduce itself in the hobbed rack or gear. This is the setting angle for a hob where the fillet is reduced to a minimum.

We will now take this same hob and set it at an angle of 3 deg. 11 min, which is very close to its helix angle at the middle of the hob tooth, and calculate the height of the fillet on the hobbed rack under these conditions. This gives us the following values:

$$K = 0.52381 = {}^{11}\!/_{21} \text{ in.}$$

$$E = 1.6667 \text{ in.}$$

$$\delta = 14 \text{ deg. } 35 \text{ min. } 20 \text{ sec.}$$

$$\beta = 3 \text{ deg. } 11 \text{ sec.}$$

$$a = 0.32031 \text{ in.}$$

$$\cos \alpha = \frac{\cos \delta}{\cos \beta} = \frac{0.96776}{0.99846} = 0.96925$$

$$\alpha = 14 \text{ deg. } 14 \text{ min. } 40 \text{ sec.}$$

$$= K \cos \beta = \frac{11 \times 0.99846}{21} = 0.5230 \text{ in.}$$

$$\sin \epsilon = \frac{a}{E} = \frac{0.32031}{1.6667} = 0.19218$$

$$\epsilon = 11 \text{ deg. } 4 \text{ min. } 50 \text{ sec.}$$

$$\tan \alpha' = \frac{\tan \alpha}{\sin \beta} = \frac{0.25386}{0.05553} = 4.57158$$

$$\alpha' = 77 \text{ deg. } 39 \text{ min. } 40 \text{ sec.}$$

$$\Delta = 90 \text{ deg.} - (\alpha' + \epsilon) = 1 \text{ deg. } 15 \text{ min. } 30 \text{ sec.}$$

Additional height of fillet $= E$ versin $\Delta = 1.6667 \times 0.00024 =$ 0.0004 in.

If the hob had a fillet of 0.0333 in. at the tip, the total height of the fillet at the root of the hobbed rack would be as follows:

Height of fillet $= 0.0333 + 0.0004 = 0.0337$ in.

The practical limit in the variation of the setting angle of this hob is about 3 deg. either side of the helix angle of the hob. More variation than this increases the height of the fillet to such an extent that there would be danger of fillet interference with the tip of the profile of the mating gear tooth. This limited amount of variation is enough, however, for all practical purposes.

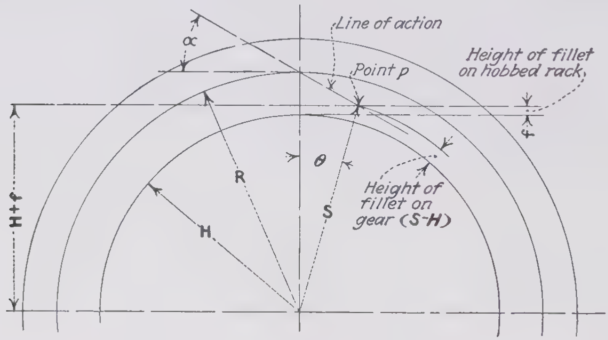


FIG. 163.—Determination of fillet conditions at root of a hobbed gear.

We will now examine the fillet conditions at the root of a hobbed gear. Here, we are concerned primarily with its height. Referring to Fig. 163, we have:

R = generating or pitch diameter when meshing with hobbed rack
H = root radius of gear
N = number of teeth in gear
α = pressure angle of hobbed rack
P = circular pitch of hobbed rack

f = height of fillet on hobbed rack
S = radius to top of fillet on hobbed gear
Θ = angle between intersection of top of fillet with line of action and center line of gear

Solving this simple problem in trigonometry, we have

$$\tan \Theta = \frac{R - (H + f)}{(H + f) \tan \alpha} \tag{148}$$

$$S = \frac{H + f}{\cos \theta} \tag{149}$$

$$\text{Height of fillet on gear} = S - H \tag{150}$$

As a definite example we will consider a 36-tooth gear hobbed with the 6-d.p. hob set square with the axis of the gear, which was used in a preceding example of a hobbed rack. From this preceding example, we get the following values:

$N = 36$
$H = 3.0000 - 0.2000 = 2.8000$ in.
$\alpha = 14$ deg. $+ 35$ min. 20 sec.
$P = 0.52381 = 11/21$ in.
$f = 0.0644$ in.

Whence

$$R = \frac{N \times P}{2\pi} = \frac{36 \times 11}{2 \times 21 \times 3.1416} = 3.0012 \text{ in.}$$

$$\tan \delta = \frac{3.0012 - (2.8000 + 0.0644)}{2.8644 \times 0.26027} = 0.18350$$

$$\Theta = 10 \text{ deg. } 23 \text{ min. } 55 \text{ sec.}$$

$$S = \frac{2.8644}{0.98358} = 2.9122 \text{ in.}$$

Height of fillet on gear $= 2.9122 - 2.8000 = 0.1122$ in.

If this fillet is so high that it extends into the active profile of the gear, interference with the tip of the mating tooth profile will take place.

Cutting Flutes and Relief of Hobs.—Thus far, we have considered only the cutting edges of gear hobs. These edges must lie in an involute helicoidal surface. In order to be effective cutting tools, the material back of these cutting edges must be relieved. The correct forms of these relieved surfaces depend upon the amount of relief and the character of the cutting flutes, that is, whether they are straight or helical, radial or undercut.

We will first examine the relieved surfaces of hobs with straight radial flutes.

Whenever the correct profile of the hob in an axial section is a very close approximation to a straight line (which occurs when the helix angle of the hob is small), the ordinary radial relief is nearly enough correct to use. In all other cases, such a relief will introduce errors such that the gears cut when the hob is new will be different from those cut after the hob has been ground back appreciably. This error can be reduced and practically eliminated by making the relief surfaces of involute helicoidal form. Incidentally, if these relieved surfaces are ground, the involute helicoid is the simplest one to maintain to any specified degree of accuracy.

In effect, the introduction of relief on the sides of the hob teeth results in making one side of the relieved surface with a slightly greater lead than that of the cutting edges, while the other side has a correspondingly lesser lead. The intersection of these two different helicoidal surfaces with the surface of the cutting flutes should be identical with the intersection of the correct (theoretical) involute helicoidal surface of the cutting edges with this same surface of the cutting flutes. In general, where the relieved surfaces are of involute helicoidal form, if the tangents of these intersection curves are identical to those of the helicoidal surfaces of the cutting edges, these intersection forms will be practically identical.

The surface of a straight radial cutting flute lies in the axial plane of the hub. Without relief, the correct profile would be developed by setting a cutting tool with an angle equal to that of the generatrix off center an amount equal to the radius of the base cylinder of the helicoid. In Fig. 164, this is illustrated.

The same conditions can also be maintained when grinding such surfaces with a cone-shaped grinding wheel. In this case, the axis of the wheel is set at any convenient angle in the horizontal plane and trued in position to the angle of the generatrix. This wheel is also set off center the same amount as for the cutting tool. In Fig. 165, this is illustrated.

When involute helicoidal surfaces are used for the relieved surfaces, the radius of the base cylinder and the angle of the generatrix for these surfaces with changed leads, one greater and the other less than that of the cutting edges, can be calculated. These two values give all the data necessary to set these

relieving tools properly. Due to the horizontal contact line between these tools and the hob surfaces, a straight-in relieving action does not make it change its horizontal position; hence, the true involute helicoidal surfaces will be produced. Also, for the same reason, the diameter of a grinding wheel thus located has no effect on the form produced. With any other setting where the contact line is not horizontal, or on any type of helicoid except the involute helicoid, a change in the diameter of the grinding wheel will also change the form of helicoid produced.

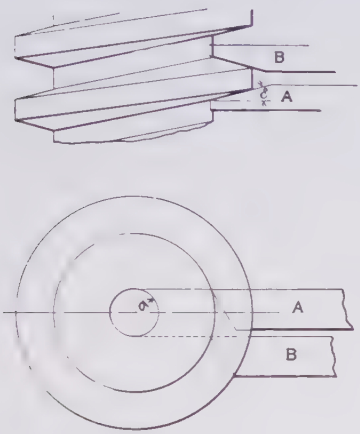


FIG. 164.

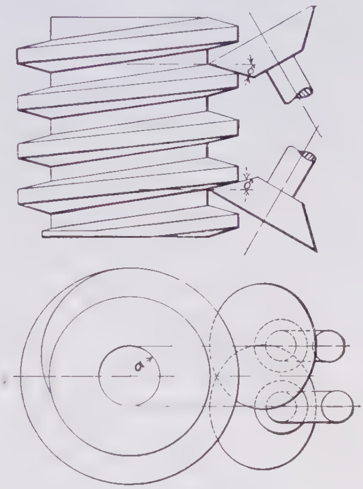


FIG. 165.

FIG. 164.—Setting of cutting tool for hob without relief.
FIG. 165.—Grinding a hob without relief with a cone-shaped wheel.

We will now determine the change in lead on the relieved surfaces of a hob. Considering first one with straight radial cutting flutes, we have

When K = lead of hob
α = pressure angle of basic rack
α'' = angle of tangent to profile on axial section at R
R = pitch radius of hob
β = helix angle of R
N = number of cutting flutes
F = radial relief of hob per flute

The leads of the relieved surfaces of the hob teeth would then be as follows:

$$\tan \alpha'' = \frac{\tan \alpha}{\cos \beta} \qquad \text{(see Eq. (127))}$$

$$\text{Greater lead} = K + NF \tan \alpha'' \tag{151}$$

$$\text{Lesser lead} = K - NF \tan \alpha'' \tag{152}$$

We will now determine the radius of the base cylinder and the angle of the generatrix of the involute helicoids with these changed leads, assuming that the tangents (at the pitch radius) to the curves formed by the intersection of these helicoids with the straight radial surfaces of the cutting flutes are the same as the tangents to the theoretical cutting edges at the same point. Thus, when

K' = greater lead
K'' = lesser lead
α_1 = basic rack for greater lead
α_2 = basic rack for lesser lead
β' = helix angle on greater lead at radius R
β'' = helix angle on lesser lead at radius R
δ' = angle of generatrix on greater lead
δ'' = angle of generatrix on lesser lead
a' = radius of base cylinder for greater lead
a'' = radius of base cylinder for lesser lead

It must be remembered that the circular pitch of these relieved surfaces remains unchanged because of the jumping action of the relieving tool. Also, that α'' is to be the same for all surfaces. We have, therefore,

$$\tan \beta' = \frac{K'}{2\pi R} \quad \text{and} \quad \tan \beta'' = \frac{K''}{2\pi R} \tag{153}$$

$$\tan \alpha_1 = \tan \alpha'' \cos \beta' \text{ and } \tan \alpha_2 = \tan \alpha'' \cos \beta' \tag{154}$$

$$\cos \delta' = \cos \alpha_1 \cos \beta' \text{ and } \cos \delta'' = \cos \alpha_2 \cos \beta'' \tag{155}$$

$${}' = \frac{K'}{2\pi \tan \delta'} \quad \text{and} \quad a'' = \frac{K''}{2\pi \tan \delta''} \tag{156}$$

As a definite example, we will consider a 6-d.p. hob, 14½-deg. pressure angle, 3-in. pitch diameter, with 15 straight radial cutting flutes, and with 0.050-in. radial relief (top rake) per flute. This gives us the following values:

$$P = 0.5236 \text{ in.}$$
$$R = 1.800 \text{ in.}$$
$$\alpha = 14\tfrac{1}{2} \text{ deg.}$$

$$\sin \beta = \frac{2\pi R}{P} = \frac{M}{2R} = \frac{1}{6 \times 2 \times 1.500} = 0.05556$$

$$\beta = 3 \text{ deg. } 11 \text{ min. } 5 \text{ sec.}$$

$$N = 15 \text{ flutes}$$

$$F = 0.050 \text{ in. per flute}$$

$$K = \frac{P}{\cos \beta} = \frac{0.5236}{0.99845} = 0.52441 \text{ in.}$$

$$\tan \alpha'' = \frac{\tan \alpha}{\cos \beta} = \frac{0.25862}{0.99845} = 0.25902$$

$$\alpha'' = 14 \text{ deg. } 31 \text{ min. } 15 \text{ sec.}$$

$$\cos \delta = \cos \alpha \cos \beta = 0.96815 \times 0.99845 = 0.96665$$

$$\delta = 14 \text{ deg. } 50 \text{ min. } 15 \text{ sec.}$$

$$a = \frac{K}{2\pi \tan \delta} = \frac{0.52441}{2 \times 3.1416 \times 0.26491} = 0.31503 \text{ in.}$$

$$K' = K + NF \tan \alpha'' = 0.52441 + (15 \times 0.050 \times 0.25902) = 0.71868 \text{ in.}$$

$$K'' = K - NF \tan \alpha'' = 0.52441 - (15 \times 0.050 \times 0.25902) = 0.33014 \text{ in.}$$

Considering the greater lead, we have

$$\tan \beta' = \frac{K}{2\pi R} = \frac{0.71868}{2 \times 3.1416 \times 0.26491} = 0.07625$$

$$\beta' = 4 \text{ deg. } 21 \text{ min. } 37 \text{ sec.}$$

$$\tan \alpha_1 = \tan \alpha'' \cos \beta' = 0.25902 \times 0.99711 = 0.25827$$

$$\alpha_1 = 14 \text{ deg. } 28 \text{ min. } 53 \text{ sec.}$$

$$\cos \delta' = \cos \alpha_1 \cos \beta' = 0.96824 \times 0.99711 = 0.96544$$

$$\delta' = 15 \text{ deg. } 6 \text{ min. } 30 \text{ sec.}$$

$$a' = \frac{K'}{2\pi \tan \delta'} = \frac{0.71868}{2 \times 3.1416 \times 0.26998} = 0.42366 \text{ in.}$$

Considering next the lesser lead, we have

$$\tan \beta'' = \frac{K}{2\pi R} = \frac{0.33014}{6.2832 \times 1.5} = 0.03503$$

$$\beta'' = 2 \text{ deg. } 0 \text{ min. } 23 \text{ sec.}$$

$$\tan \alpha_2 = \tan \alpha'' \cos \beta'' = 0.25902 \times 0.99939 = 0.25886$$

$$\alpha_2 = 14 \text{ deg. } 30 \text{ min. } 46 \text{ sec.}$$

$$\cos \delta'' = \cos \alpha_2 \cos \beta'' = 0.96809 \times 0.99939 = 0.96750$$

$$\delta'' = 14 \text{ deg. } 38 \text{ min. } 53 \text{ sec.}$$

$$a'' = \frac{K''}{2\pi \tan \delta''} = \frac{0.33014}{6.2832 \times 0.26136} = 0.20103 \text{ in.}$$

To capitulate, referring to Figs. 164 or 165, we have the following:

	Cutting edges	Greater lead	Lesser lead
Angle of tool	14° 50′ 15″	15° 6′ 30″	14° 38′ 53″
Distance off center measured vertically, inches	0.31503	0.42366	0.20103

In all these examples, as noted before, we will deal only with the tangent to the profiles at the pitch radius. In the case of straight radial flutes, this tangent remains constant regardless of the amount of relief or number of cutting flutes. This tangent was at the angle α'' in the preceding examples.

When the gear hob has helical cutting flutes, usually normal to the helix of the cutting edges at the pitch diameter, the axial section of the relieved surfaces of the hob shows a distortion in form. In addition, with helical cutting flutes, the equations for the greater and lesser leads of the relieved surfaces are different from those for straight flutes. Thus, when

R = pitch radius of hob
α'' = angle of tangent in axial section to helicoid of the cutting edges at R
L = lead of cutting flutes
ϵ = helix angle of cutting flutes at radius R measured from the axis of the hob

$$\tan \epsilon = \frac{2\pi R}{L} \tag{157}$$

$$\text{Greater lead} = K + NF \cos \epsilon \tan \alpha'' \tag{158}$$

$$\text{Lesser lead} = K - NF \cos \epsilon \tan \alpha'' \tag{159}$$

The axial section of the relieved surfaces, due to the relief and the helix of the cutting flutes, will be distorted at the tip and on the sides of the hob tooth. Referring to Fig. 166, let

Δ = distortion angle at tip
Δ' = angle on side of greater lead
Δ'' = angle on side of lesser lead

Then,

$$\tan \Delta = \frac{NF}{L} \tag{160}$$

$$\cot \Delta' = \cot \alpha'' - \tan \Delta \tag{161}$$

$$\cot \Delta'' = \cot \alpha'' + \tan \Delta \tag{162}$$

As a definite example, we will take a 6-d.p. hob, 14½-deg. pressure angle, 3-in. pitch diameter, double threaded with helical cutting flutes normal to the helix of the cutting edges of the pitch diameter. This hob will have 15 cutting flutes and a radial relief

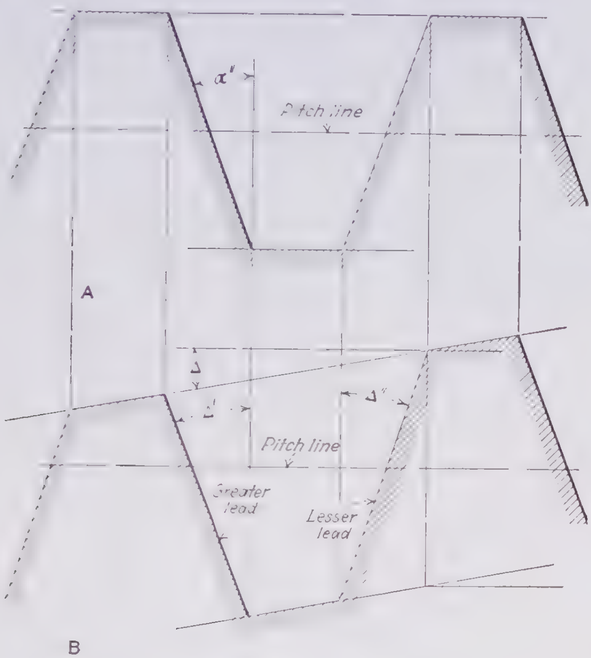


FIG. 166.—*A*, Axial section of helicoid containing cutting edges. *B*, Axial section of relieved surfaces.

of 0.200 in. per flute. This relief is much greater than would be used in practice, but it is used here to bring out more prominently the nature of the distortion. This gives us the following values:

$$R = 1.500 \text{ in.}$$
$$\alpha = 14\tfrac{1}{2} \text{ deg.}$$
$$P = 0.5236 \text{ in.}$$
$$M = \tfrac{1}{6}$$
$$N = 15 \text{ flutes}$$
$$F = 0.200 \text{ in. per flute}$$
$$\sin \beta = \frac{2M}{2R} = 0.1111$$
$$\beta = 6 \text{ deg. } 22 \text{ min. } 46 \text{ sec.}$$

$$
\begin{aligned}
K &= \frac{2P}{\cos \beta} = 1.05372 \text{ in.} \\
L &= \frac{2\pi R}{\tan \beta} = \frac{3 \times 3.1416}{0.11180} = 84.30034 \text{ in.} \\
\tan \alpha'' &= \frac{\tan \alpha}{\cos \beta} = \frac{0.25862}{0.99381} = 0.26023 \\
\alpha'' &= 14 \text{ deg. } 35 \text{ min. } 12 \text{ sec.} \\
\tan \Delta &= \frac{NF}{L} = \frac{15 \times 0.2000}{84.30034} = 0.03559 \\
\Delta &= 2 \text{ deg. } 2 \text{ min. } 18 \text{ sec.} \\
\cot \Delta' &= \cot \alpha'' - \tan \Delta = 3.84274 - 0.03559 = 3.80715 \\
\Delta' &= 14 \text{ deg. } 43 \text{ min. } 2 \text{ sec.} \\
\cot \Delta'' &= \cot \alpha' + \tan \Delta = 3.84274 + 0.03559 = 3.87833 \\
\Delta'' &= 14 \text{ deg. } 27 \text{ min. } 30 \text{ sec.}
\end{aligned}
$$

The calculation of the angle and setting of the relieving tool is made in the same manner as for straight radial cutting flutes, except that the angle Δ' is substituted for α'' in the second equation for the greater lead, and the angle Δ'' is substituted for α'' in the second equation for the lesser lead. This gives us the following:

$$
\begin{aligned}
\tan \beta' &= \frac{K'}{2\pi R} \qquad \text{and } \tan \beta'' = \frac{K''}{2\pi R} \\
\tan \alpha_1 &= \tan \Delta' \cos \beta' \text{ and } \tan \alpha_2 = \tan \Delta'' \cos \beta'' \\
\cos \delta' &= \cos \alpha_1 \cos \beta' \text{ and } \cos \delta'' = \cos \alpha_2 \cos \beta'' \\
a' &= \frac{K'}{2\pi \tan \delta'} \qquad \text{and} \qquad a'' = \frac{K''}{2\pi \tan \delta''}
\end{aligned}
$$

Thus far, we have considered radial cutting flutes only. The use of an undercut flute introduces further distortions into the form of the tooth on the hob. Undercut flutes are seldom used on hobs for gears, for several reasons. In the first place, an undercut flute makes a weak cutting edge at the tip of the hob tooth which soon breaks down if the hob is used for roughing, because the tip of the hob tooth takes the heaviest cuts. On the other hand, if the hob is used only for finishing the sides of the gear-tooth profile, an undercut flute helps produce a smoother cut. A second disadvantage of an undercut flute is the greater difficulty of sharpening it properly, particularly if the flutes are helical. A third disadvantage is the difficulty usually experienced in correcting the form of the relieved surfaces to compen-

sate for the undercut flute. The curve of the intersection of an involute helicoid with a plane parallel to its axis is not symmetrical unless the plane actually passes through the axis of the helicoid. With a straight flute which is undercut, we have the conditions shown in Fig. 167, where

h = distance of flute or intersecting plane from axis of involute helicoid

FIG. 167.—Hob with undercut flutes.

The equation of the intersection curve of an involute helicoidal surface with such a plane is as follows:

$$x = \frac{K}{2\pi}\left(\frac{\sqrt{h^2 - a^2 + y^2}}{a} - \cos^{-1}\frac{ay \pm h\sqrt{h^2a^2 + y^2}}{h^2 + y^2}\right) \qquad (163)$$

The equation of the tangent to this curve measured from the x-axis is as follows:

$$\tan\,\phi = \frac{dy}{dx} = \frac{2\pi a(h^2 + y^2)}{K(ah \pm y\sqrt{h^2 - a^2 + y^2})} \qquad (164)$$

When this intersecting plane is tangent to the base cylinder, h becomes equal to a, and the foregoing equations reduce to the following:

$$x = \frac{K}{2\pi}\left(\frac{y}{a} - \cos^{-1}\frac{ay \pm ay}{a^2 + y^2}\right) \qquad (165)$$

$$\tan\,\phi = \frac{2\pi a(a^2 + y^2)}{K(a^2 \pm y^2)} \qquad (166)$$

In this case, one side of the profile becomes a straight line while the other is curved. To show this graphically, we will plot the intersection curve of the triple-threaded hob used in a previous example with a plane tangent to its base cylinder. This gives us the following values:

$$K = 1.59309 \text{ in.}$$
$$a = 0.81256 \text{ in.}$$
$$R = 1.5000 \text{ in.}$$

	One side		Opposite side	
y	x	$x-0.46805$	x	$x-0.32153$
1.3333	0.41605	−0.05200	0.29532	−0.02621
1.3667	0.42645	−0.04160	0.30019	−0.02134
1.4000	0.43685	−0.03120	0.30525	−0.01628
1.4333	0.44725	−0.02080	0.31050	−0.01103
1.4667	0.45765	−0.01040	0.31594	−0.00559
1.5000	0.46805	0	0.32153	0
1.5333	0.47845	+0.01040	0.32730	+0.00577
1.5667	0.48885	+0.02080	0.33321	+0.01168
1.6000	0.49925	+0.03120	0.33927	+0.01774
1.6333	0.50965	+0.04160	0.34548	+0.02395
1.6667	0.52005	+0.05200	0.35182	+0.03029

The tangent to these curves at the pitch line becomes

$$\tan \phi \text{ (one side)} = \frac{2\pi a}{K} = \frac{1}{\tan \delta} = 3.20476$$
$$\phi = 72 \text{ deg. } 40 \text{ min. } 13 \text{ sec.}$$
$$\tan \phi \text{ (opposite side)} = \frac{2\pi a(a^2 + R^2)}{K(R^2 - a^2)} = 5.86677$$
$$\phi = 80 \text{ deg. } 19 \text{ min. } 37 \text{ sec.}$$

These values are plotted in Fig. 168. The opposite side is plotted in the reverse direction along the X-axis, so as to show the form of the tooth space of the hob.

The calculation for the angle and setting of a relieving tool for this type of cutting flute is too long and complex to be included here. When the helix angle of the hob is small, however, and the axial section of the involute helicoid of the cutting edges is a close approximation to a straight line, a simple solution is available, since a straight-sided tool set at the center of the hob can produce a satisfactory radial-flute hob. To produce such a hob with undercut flutes, a close approximation to the correct form

of the relieved surfaces can be obtained by setting a suitable distance off center the same tool that would be used to produce the radial-flute hob. This distance would be determined for straight undercut flutes as follows:

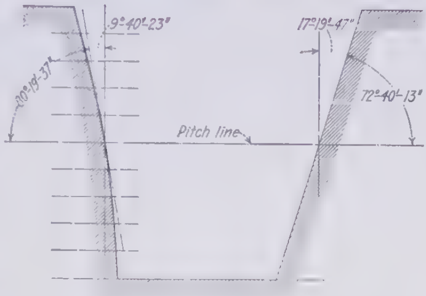


FIG. 168.—Intersection of helicoidal surfaces of a relieved hob with a plane tangent to the base cylinder.

When R = pitch radius of hob
N = number of flutes
F = radial relief of hob per flute
h = distance of undercut flute off center
γ = angle of radial relief
γ' = angle of undercut flute with center line of hob
γ'' = angle to tip tool to compensate for undercut flutes
h' = distance off center to set relieving tool

$$\tan \gamma = \frac{NF}{2\pi R} \tag{167}$$

$$\sin \gamma' = \frac{h}{R} \tag{168}$$

$$\cos (\gamma + \gamma'') = \cos \gamma \, (1 - \tan \gamma \tan \gamma') \tag{169}$$

$$h' = R \sin \gamma'' \tag{170}$$

The setting of such a tool to produce a similar hob with helical undercut flutes would be the same as the foregoing, except that

$$\tan \gamma = \frac{NF \cos \epsilon}{2\pi R} \tag{171}$$

Where, ϵ = helix angle of flute at radius R measured from the axis of the hob
L = lead of flute

$$\tan \epsilon = \frac{2\pi R}{L} \qquad \text{(see Eq. (157))}$$

As a definite example, we will take a hob, 3 in. in pitch diameter, having 15 straight flutes with 0.100-in. radial relief per flute, and 0.300-in. undercut. The radial relief here is greater than would be used in actual practice, but it is used in this example to bring out more clearly the change in the setting of the tool required to compensate for it. This gives the following values:

$$
\begin{aligned}
R &= 1.500 \text{ in.} \\
N &= 15 \text{ flutes} \\
F &= 0.1000 \text{ in. per flute} \\
h &= 0.3000 \text{ in.} \\
\tan \gamma &= \frac{15 \times 0.1000}{2 \times 3.1416 \times 1.500} = 0.15915 \\
\gamma &= 9 \text{ deg. } 2 \text{ min. } 34 \text{ sec.} \\
\cos \gamma &= 0.98757 \\
\sin \gamma' &= \frac{0.300}{1.5000} = 0.2000 \\
\gamma' &= 11 \text{ deg. } 32 \text{ min. } 13 \text{ sec.} \\
\tan \gamma' &= 0.20412 \\
\cos (\gamma + \gamma'') &= 0.98757(1 - 0.15915 \times 0.20412) \\
&= 0.95549 \\
(\gamma + \gamma'') &= 17 \text{ deg. } 9 \text{ min. } 30 \text{ sec.} \\
\gamma'' &= 8 \text{ deg. } 6 \text{ min. } 56 \text{ sec.} \\
\sin \gamma'' &= 0.14117 \\
h' &= 1.500 \times 0.14117 = 0.2118 \text{ in.}
\end{aligned}
$$

In this example, the same tool that would be set on center to produce a radial-flute hob would be set 0.2118 in. off center in the direction of the undercut to produce the undercut-flute hob.

It must be apparent that great care should be exercised in resharpening a hob. If the flutes as ground are not accurately indexed, or if the flute surface is not maintained in its proper relationship to the axis of the hob, the cutting edges will not remain in the surface of the helicoid to which they belong. The original accuracy of the hob may thus be lost entirely by carelessness in resharpening.

Cutting Action of Hobs.—Hobbing is entirely different from milling. The only common feature of the two is that both are continuous cutting operations. The cutting contact of the hob with a gear blank is comparatively short, and the cutting resistance is very unevenly distributed.

When the first roughing cut is made, most of the cutting takes place on the entering side of the gear blank. By the time a tooth space has reached the vertical center line of the blank in relation to the hob, most of the metal has been removed. There are three cutting edges on each tooth of the hob: the tip and the two sides. The cutting edge at the tip is not only the shortest but it also removes about one-half of the metal. Even so, it is not

always cutting continuously. The amount of time that the cutting edge at the tip is in operation depends largely upon the number of teeth in the gears being produced. Thus, when hobbing a gear with 25 teeth, this edge will cut during half a revolution only. With fewer teeth in the gear, the relative time will be less; with more teeth in the gear, the time will be greater. This edge does not cut during a full revolution until the number of teeth in the gear is about 200. The cutting edges on the sides of the hob teeth do not cut continuously, either. First one side will be cutting, then both sides together, and finally the opposite side only. The duration of the cutting on the sides of the hob teeth also depends largely upon the number of teeth in the hobbed gear and is considerably shorter than that on the cutting edges at the tip. In addition, the tendency of these side-cutting edges of the hob teeth at one instant is to drive the gear blank ahead and at another instant to hold it back. This combination of irregular loads, both in amount and direction, tends to make a difficult production operation even more difficult. In the first place, the hobbing machine must be sufficiently rigid and massive to withstand these loads and absorb or dampen the vibrations set up by the irregular cutting action of the hob. The requirements of hobbing machines in this respect are far more exacting than those of plain milling machines. In the second place, the hob also has a much more severe duty to perform than a milling cutter, so that its cutting edges will break down under finer feeds and lower speeds than a milling cutter will. For all these reasons, as noted before, in order to reproduce the maximum of the accuracy of the hobbing machine, individual finishing cuts must be taken on each side of the gear tooth. The surface of the tooth profile of a hobbed gear is composed of a series of hills and valleys. The true involute surface lies at the bottom of the valleys. The amount of metal projecting above this theoretically true surface depends upon the diameter of the hob, the number of cutting flutes in the hob, and its feed; also, upon the number of teeth in the hobbed gear. The larger the diameter of the hob, and the greater the number of cutting flutes, and the finer the feed, and the larger the number of teeth in the gear, the smaller will be the amount of material left projecting above the true involute surface. When the amount of this excess metal is not excessive, these high points will break down quickly when the gears are operated under load, either in service or as a

smoothing or burnishing operation, machines for performing which are now on the market.

All production processes have their advantages and limitations, and in order to obtain full advantage of any process, its limitations must be fully appreciated. The first limitation of the hobbing process that we shall consider is the cutting speed of the hob. This depends largely upon the characteristics of the material being cut and the physical characteristics of the material in the hob. As noted before, this cutting speed should be somewhat less than that for a milling cutter under similar conditions. After the maximum cutting speed has been established, the next step is to determine how to take the greatest advantage of this cutting speed. Here we must consider the diameter of the hob. The smaller the diameter of this hob, the more revolutions per unit of time it will make at a given cutting speed, and the faster the gear blanks will revolve. Considering this point only, the smaller the diameter of the hob, the faster the production will be. But the smaller the diameter of the hob, the fewer the cutting flutes, and the finer the feed must be in order to obtain a sufficiently smooth tooth profile. In addition, the smaller the diameter of the hob, the more sensitive the correct hob-tooth profile will be and the greater the difficulty of producing accurate hobs. This means that the final answer must be a compromise between these opposing factors.

One possible solution is the development of more accurate multiple-threaded hobs. With the same diameter and cutting speed for the hob, the gear blank will revolve twice as fast with a double-threaded hob as with a single-threaded hob; three times as fast with a triple-threaded hob; etc. To obtain the desired smoothness of surface, the feed would have to be reduced slightly for a multiple-threaded hob. Also, the number of flutes in these multiple-threaded hobs should not be a multiple of the number of threads or starts on the hob. The reduction of feed, however, would not be proportional to the increased rate of rotation of the gear blanks, so that multiple-threaded hobs offer opportunities of much faster production. Such hobs are now widely used for roughing operations. Satisfactory finishing cuts under these conditions, however, can be secured only at the expense of more accurate multiple-threaded hobs. These would have more sensitive tooth profiles than single-threaded ones, and a straight line approximation would not be sufficiently accurate.

CHAPTER XI

SHAPING OF GEAR TEETH

Another general process used for the production of gear teeth consists of shaping or planing the tooth forms. The earlier machines of this type used a form tool of the shape of the tooth space or a standard shaping tool with a template to control its position. These methods are used today, chiefly on very large gears. Where a template type of shaping machine is used, a series of templates corresponding to a series of formed milling cutters can be used, and corrections in the settings of these templates to compensate for any theoretical error in the form can be made exactly as previously described for formed milling cutters. The same is also true when a formed shaping tool is employed. Fundamentally, these two shaping methods are identical to form milling of gears. The only difference is that a shaping tool is substituted for the formed milling cutter.

Another method of shaping gear teeth consists of a molding or generating process on the principle of two gears in mesh, where one of the gears is a reciprocating cutter while the other is the gear blank that is being molded to form. This process was developed by the Fellows Gear Shaper Company. In Fig. 169, one of these machines is illustrated.

The gear blank is held on an arbor while the cutter is mounted on a spindle carried in the reciprocating ram of the machine. The cutter and work spindles are connected by means of gearing, and each is provided with an index wheel and worm. Change gears are used to obtain the proper relation between the number of teeth in the cutter and the number of teeth in the gear being cut. In operation, the cutter and work rotate slowly together, thus producing on the gear blank the conjugate form to the teeth on the cutter.

Here, as with hobbing, the accuracy of the product depends upon three main factors: the accuracy of the machine, the accuracy of the cutter, and the carefulness of the operator in mounting the gear blanks and in resharpening the cutters.

Here, again, eccentricity of change gears, index wormwheels, cutters, etc., will be reproduced on the product in the form of troublesome tooth-profile or spacing errors. Concentricity of the cutter and cutter spindle is of particular importance here. If eccentricity or cumulative spacing errors in the teeth of the cutter should exist, in those cases where the number of teeth in the gear produced is the same as the number of teeth in the cutter, the result on the product will be the same as a similar amount

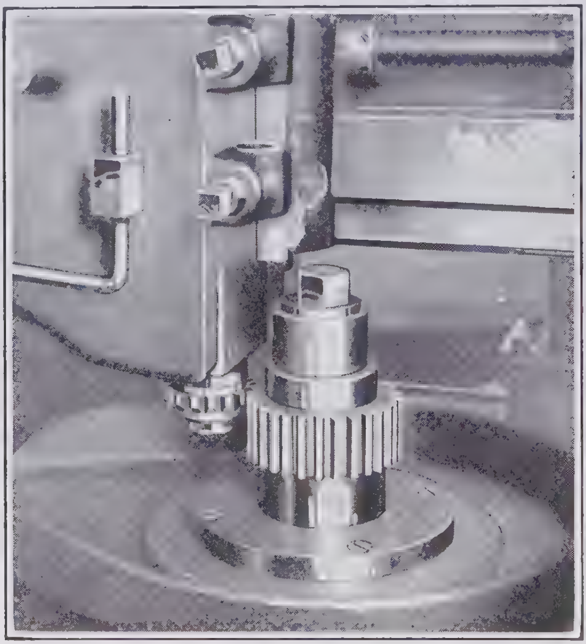


Fig. 169.—The Fellows gear shaper in action.

of eccentricity there. If the gear has a different number of teeth than the cutter, sometimes an excessive spacing error will be produced between the first and last tooth space cut because of the cumulative effect of such errors. This local spacing error will be there in addition to the eccentric error effect also present. Thus, as with hobbing, the accuracy of the product depends largely upon the accuracy of the cutters. The machine factors, such as rigidity, alignment, lost motion because of backlash in gear trains, etc., are very similar to those of the hobbing machines.

We will therefore direct our attention to the problem of these pinion-shaped cutters.

In the first place, the diameter of these cutters is limited. They are usually made 3 or 4 in. in pitch diameter, the 3-in. cutter being the one most extensively used. In Fig. 170, such a cutter is shown.

This method of shaping teeth has the same advantage as hobbing, namely, that one cutter will generate mating gears of any number of teeth. The limited diameter of the cutter, however, introduces a serious handicap. For coarser pitches, the number of teeth in the cutter is so small that the tooth profile of the cutter extends below the base circle of the involute so that a full involute tooth form cannot be produced on the product. Usually, such cutters have straight flanks below the base circle, approximately radial in direction, and this modification affects the tips of the teeth of the product. In addition, the small diameter of the cutter results in a high fillet, so that gears cut by this process need much more clearance at the root than those produced by other methods. These limitations were largely responsible for the wide use of the 20-deg. stub-tooth system with this process, as this tooth form lends itself to this method of production better than any other standard tooth form.

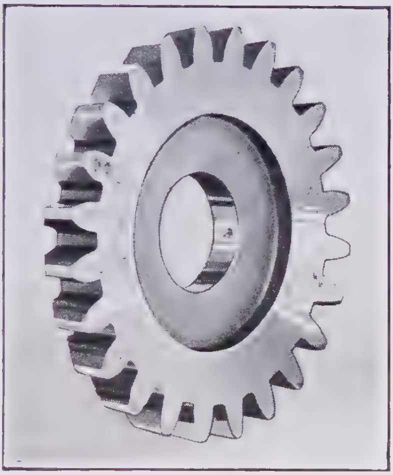


Fig. 170.—A Fellows gear-shaping cutter of 3-in. pitch diameter.

We will first direct our attention to the form of the rack that will be conjugate to a pinion-shaped cutter with a radial flank below the base circle, studying primarily the form of the fillet at the root of this rack tooth and the modification at its tip.

Referring to Fig. 171, let

E = outside radius of pinion-shaped cutter
R = pitch diameter
ϵ = angle of rotation of cutter

The equation of the fillet developed by the corner of the tooth of the cutter would be as follows:

$$x = E \sin \epsilon - R\epsilon \tag{172}$$
$$y = R - E \cos \epsilon \tag{173}$$

This curve is related to a cycloid. If the point that traces the curve were on the pitch circle, it would be a cycloid. The equation of the tangent to this curve is as follows:

$$\tan \phi = \frac{dy}{dx} = \frac{E \sin \epsilon}{E \cos \epsilon - R} = \frac{E \sin \epsilon}{-y} \tag{174}$$

When x' and y' are the coordinates for the line of action, the equation of the line of action will be as follows:

$$x' = -y \tan \phi = E \sin \epsilon \tag{175}$$
$$y' = y \tag{176}$$

The line of action in this case is the outside circle of the cutter. The tangent to the fillet where it crosses the pitch line is perpendicular to the X-axis. The theoretical contact would shift at this point from the outside surface of the fillet to the inside surface. Or, in other words, the conjugate action on the surface of the fillet below the pitch line would be as a spur gear, while the action on the surface of the fillet above the pitch line would be as an internal gear. Conjugate action is theoretically possible on this fillet, but it is hardly practical.

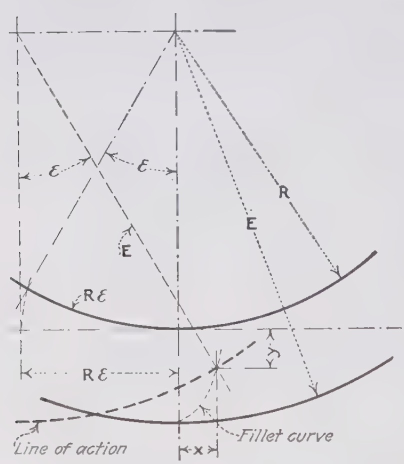


FIG. 171.—Development of a rack formed by a pinion-shaped cutter.

We will now consider the form produced by the radial flank of the cutter. This would be of cycloid form, with a rolling circle equal to one-half the pitch diameter of the cutter, which would give us the following equation for the curve at the tip of the conjugate-rack tooth:

$$x = \frac{R}{2}(2\epsilon - \sin 2\epsilon) \tag{177}$$

$$y = \frac{R}{2}(1 - \cos 2\epsilon) \tag{178}$$

$$\tan \phi = \frac{dy}{dx} = \frac{\sin 2\epsilon}{1 - \cos 2\epsilon} \tag{179}$$

For the line of action, we should have

$$x' = -y \tan \phi = \frac{R}{2} \sin 2\epsilon \tag{180}$$

$$y' = y \tag{181}$$

This line of action for a cycloid, as we have seen before, is the outline of the rolling circle.

As a definite example, we will take a 3-in. pitch diameter pinion-shaped cutter, 4 d.p., 12 teeth, 20-deg. pressure angle, and determine the profile and line of action of its conjugate or basic rack. This gives us the following values.

Outside radius	$= E$	$= 1.8125$ in.
Pitch radius	$= R$	$= 1.5000$ in.
Pressure angle	$= \alpha$	$= 20$ deg.
Base-circle radius	$= a$	$= 1.4095$ in.

We know that the part of the basic-rack tooth profile that meshes with the involute portion of the pinion-shaped cutter will be a straight line at 20 deg. from the center line of the cutter. The fillet at the root and the cycloid at the tip will start where the tangents to these two curves are also at 20 deg. from this center line. Solving the foregoing equations for these curves, we get:

ϵ, degrees	Profile of fillet		Line of action		
	x	y	$\tan \phi$	x'	y'
0	0	−0.31250	0	0	−0.31250
5	0.02707	−0.30560	0.51692	0.15797	−0.30560
10	0.05294	−0.28496	1.10450	0.31474	−0.28496
15	0.07641	−0.25074	1.87090	0.46911	−0.25074
20	0.09631	−0.20319	3.05090	0.61991	−0.20319
25	0.11150	−0.14268	5.36863	0.76600	−0.14268
30	0.12085	−0.06967	13.62168	0.90625	−0.06967

Solving the equation for the profile of the cycloid at the tip of the rack tooth, we get the following:

ϵ, degrees	Profile of cycloid		Line of action		
	x	y	tan ∞ ϕ	x'	y'
0	0	0	....	0	0
5	0.00066	0.01139	11.43005	−0.13024	0.01139
10	0.00528	0.04523	5.67128	−0.25652	0.04523
15	0.01770	0.10048	3.73205	−0.37500	0.10048
20	0.04151	0.17547	2.74748	−0.48209	0.17547
25	0.07997	0.26791	2.44298	−0.57453	0.26791
30	0.13588	0.37500	1.73205	−0.64952	0.37500

These values are plotted in Fig. 172, which shows the profile and line of action on a rack that is conjugate to this cutter. We have seen before that, in order for a basic rack to be a suitable one for interchangeable gears, its line of action must be symmetrical about the pitch point. The line of action in this

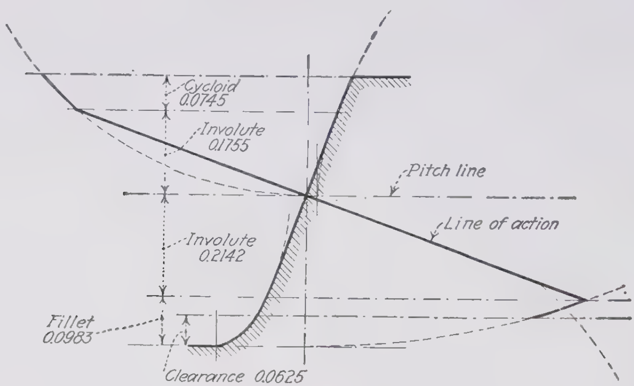


FIG. 172.—Profile of a rack conjugate to a pinion-shaped cutter of 3-in. pitch diameter, 4 d.p., 20-deg. pressure angle.

case is not symmetrical. It will be noted that more metal is left in the fillet than is removed at the tip, so that there would be tip interference when the active profile of a gear extends into the fillet. This example uses a pitch coarser than is generally used on these machines. With a 14½-deg. pressure angle, both the fillet at the root of the rack tooth and the cycloid at its tip would cover a greater part of the tooth profile, which, in turn, would result in more interference.

We will now determine the height of the fillet at the root of the tooth of a gear generated by a pinion-shaped cutter. The height of this fillet will depend upon the diameter of the cutter and the diameter of the gear blank. It will extend from the root of the gear to the bottom of the active profile of the mesh of the generated gear with the cutter. Referring to Chapter II, for the active profile of the gear, we have

$$S = \sqrt{a_2^2 + [C \sin \alpha - \sqrt{E_1^2 - a_1^2}]^2}$$

Where S = radius to top of fillet or bottom of active profile of gear when meshing with cutter

a_1 = radius of base circle of cutter

a_2 = radius of base circle of gear

C = center distance

E_1 = outside radius of cutter

α = pressure angle

H = root radius of gear

Height of fillet = $S - H$.

As a definite example, we will take a 6 d.p., 3-in. pitch diameter, 20-deg. full-depth form cutter generating a 6-in. pitch-diameter gear, and determine the height of the fillet. This gives us the following:

$$\begin{aligned}
\alpha &= 20 \text{ deg.} \\
R_1 &= 1.5000 \text{ in.} \\
R_2 &= 3.0000 \text{ in.} \\
C &= 4.5000 \text{ in.} \\
a_1 &= R_1 \cos \alpha = 1.4095 \text{ in.} \\
a_2 &= R_2 \cos \alpha = 2.8190 \text{ in.} \\
E_1 &= 1.7083 \text{ in.} \\
H &= 1.7917 \text{ in.}
\end{aligned}$$

$$S = \sqrt{(2.8190)^2 + [4.500 \times 0.34202 - \sqrt{(1.7083)^2 - 1.4095)^2}]^2} = 2.8768 \text{ in.}$$

Height of fillet = $2.8768 - 2.7917 = 0.0851$ in.

If this fillet extends into the active profile of the gear when meshing with its mating gear, interference will take place. This can be determined by calculating the radius to the bottom of the active profile of the gear when meshing with its mating gear and comparing this radius with that to the top of the fillet. Thus, if

the foregoing 6-in. pitch-diameter gear were to mesh with another of the same size, we would have the following:

$$\begin{aligned} \alpha &= 20 \text{ deg.} \\ R_1 &= 3.0000 \text{ in.} \\ R_2 &= 3.0000 \text{ in.} \\ C &= 6.0000 \text{ in.} \\ a_1 &= 2.8190 \text{ in.} \\ a_2 &= 2.8190 \text{ in.} \\ E_1 &= 3.1667 \text{ in.} \end{aligned}$$

$$S = \sqrt{(2.8190)^2 + [6.000 \times 0.34202 - \sqrt{(3.1667)^2 - (2.8190)^2}]^2} = 2.8843 \text{ in.}$$

In this example, the bottom of the active profile is 0.0075 in. above the top of the fillet, so that it just avoids interference.

All of the foregoing has been based on a sharp corner at the tip of the tooth of the cutter. Actually, these corners are broken with a small radius or bevel to remove the sharp corner. This radius is kept as small as possible. The effect of it is to increase the height of the fillet at the root of the gear tooth an amount somewhat less than the radius or bevel. This can be included in the calculation of the height of the fillet, when necessary, with sufficient accuracy by reducing the figure used for the outside radius of the cutter by an amount equal to this radius or bevel.

It is possible to reduce the extent of the radial flanks on these cutters by carrying the cutting profile farther out on the involute profile, but this, in turn, will result in a higher fillet on the generated gears because of the higher pressure angle at which they would then be generated. These characteristics of pinion-shaped cutters limit their most effective use to gears of relatively fine pitches and small diameters.

RELIEVING PINION-SHAPED CUTTERS

Sides of the teeth of pinion-shaped gear cutters must be relieved to give a proper cutting edge but must retain the correct form at the proper diameter. This is accomplished by making these relieved surfaces involute helicoids, the relief on one side of the tooth being right hand, while the relief on the other side is left hand. In other words, one side of the relieved tooth is a right-hand helical gear, while the other side is a left-hand helical gear. Mathematically, these cutters are nothing more than multiple-

threaded hobs with an infinite lead, the cutting action being obtained by reciprocating the cutter instead of revolving it. The tops of the teeth are also relieved in such a way that the width at the top of the cutter tooth remains approximately the same. As the face of the cutter is ground back when it is resharpened, a different portion of the involute profile of the cutter comes into action. This, however, as we have seen before when considering the action of one involute with another, does not change the form of the gear produced.

The intersection of the side-relieved surfaces of the cutter with a plane perpendicular to its axis is an involute curve. The polar equation of this curve, as shown before, is as follows:

$$\theta = \sqrt{\left(\frac{r}{a}\right)^2 - 1} - \tan^{-1}\sqrt{\left(\frac{r}{a}\right)^2 - 1} \quad \text{(See Eq. (14))}$$

where

r = any radius
a = radius of base circle

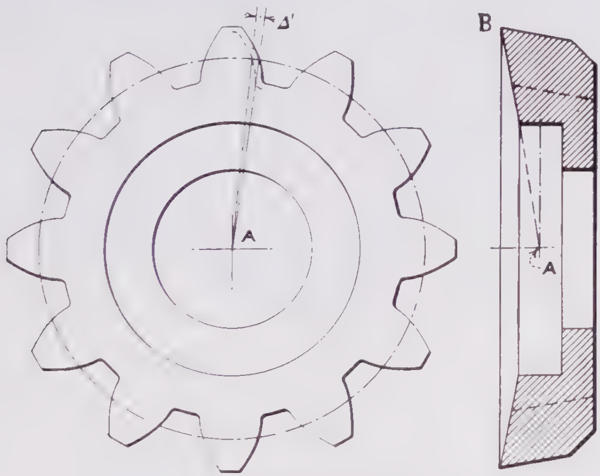


FIG. 173.—Typical pinion-shaped gear cutter.

The front face of these pinion-shaped cutters is not a plane, however, but is a conical surface. Figure 173 shows a cutter in front elevation and axial section, respectively. Line AB is the intersection line of the conical-front surface with the drawing plane. The angle of this line against a plane which is perpendicular to the axis of the cutter has been designated γ. The axial

elevation of any point of the conical surface from the apex A will be called z;
whence

$$z = r \tan \gamma$$

Involute intersection curves are obtained by intersecting the relieved-side surfaces of the cutter with planes that are perpendicular to the axis of the cutter. If these planes are equally spaced, the involute intersection curves will be twisted uniformly from some common center line, the amount of this twisting depending upon the lead of the involute-helicoidal relieved surfaces. The twisting or turning angle Δ', in Fig. 173, is dependent upon the axial distance of the intersecting plane from the apex A. Thus, when this distance is equal to the lead of the involute-helicoidal surfaces, the turning angle is equal to 2π radians.

In a fixed polar coordinate system with the origin at A, the equation of the involute that is intersected at a distance z from A becomes

$$\theta = -2\pi \frac{z}{K} + \sqrt{\left(\frac{r}{a}\right)^2 - 1} - \tan^{-1} \sqrt{\left(\frac{r}{a}\right)^2 - 1}$$

Introducing the relation,

$$z = r \tan \gamma$$

we get the following equation of the intersection curve:

$$\theta = -\frac{2\pi}{K} \tan \gamma r + \sqrt{\left(\frac{r}{a}\right)^2 - 1} - \tan^{-1} \sqrt{\left(\frac{r}{a}\right)^2 - 1}$$

Let

$$\frac{2\pi}{K} \tan \gamma = G \tag{182}$$

where

$$K = \text{lead of helix}$$

whence,

$$\theta = -Gr + \sqrt{\left(\frac{r}{a}\right)^2 - 1} - \tan^{-1} \sqrt{\left(\frac{r}{a}\right)^2 - 1} \tag{183}$$

The derivation of the tangent to this curve is as follows:

$$\frac{d\theta}{dr} = -G + \frac{\frac{r}{a^2}}{\sqrt{\left(\frac{r}{a}\right)^2 - 1}} - \frac{\frac{r}{a^2}}{\left(\frac{r}{a}\right)^2 \sqrt{\left(\frac{r}{a}\right)^2 - 1}}$$

$$-G + \frac{1}{r}\sqrt{\left(\frac{r}{a}\right)^2 - 1}$$

The inclination of the tangent against the radius is

$$\tan \psi = r\frac{d\theta}{dr}$$

The angle of this tangent at the pitch line of the cutter is equal to the pressure angle of the cutting profile. Thus, we have, when α = pressure angle of the cutting profile at the pitch line

$$\tan \alpha = -Gr + \sqrt{\left(\frac{r}{a}\right)^2 - 1} \qquad (184)$$

If this angle is to be correct at the pitch radius of the cutter, we have

α = pressure angle of cutting edges
α' = pressure angle of involute-helicoidal surfaces
a_1 = base circle of involute with pressure angle of α
a_2 = base circle of involute with pressure angle of α'
R = radius of pitch circle

$$\tan \alpha = -GR + \sqrt{\left(\frac{R}{a_2}\right)^2 - 1} = \sqrt{\left(\frac{R}{a_1}\right)^2 - 1}$$

$$\tan \alpha' = \sqrt{\left(\frac{R}{a_2}\right)^2 - 1} = \tan \alpha + GR$$

But

$$GR = \frac{2\pi R}{K} \tan \gamma$$

$\frac{2\pi R}{K}$ = tangent of the angle of the helix of the involute helicoid at the pitch radius against the axis

Let this angle be Σ, and then we have

$$GR = \tan \gamma \tan \Sigma \qquad (185)$$

$$\tan \alpha' = \tan \alpha + \tan \gamma \tan \Sigma \qquad (186)$$

We will take, as an example, a 3-in. pitch-diameter cutter for forming standard 20-deg. gears. We will assume that the angle of the relief at the top of the cutter is 7 deg., and the angle of the conical front face 5 deg. This gives the following:

R = 1.500 in.
γ = 5 deg.
α = 20 deg.

If we assume that the angle of relief Σ at the pitch line of the cutter is equivalent to the angle of the top relief, we have

$$\tan \Sigma = \tan 7 \text{ deg. } \tan \alpha$$

whence, we have

$$\begin{aligned} \tan \alpha' &= \tan \alpha + \tan \gamma \tan \Sigma = \tan \alpha\,(1 + \tan 7 \text{ deg. } \tan \gamma) \\ &= 0.36397\,(1 + 0.01074) = 0.36788 \\ \alpha' &= 20 \text{ deg. } 11 \text{ min. } 51 \text{ sec.} \end{aligned}$$

The profile of the cutting edge of a pinion-shaped cutter with a conical-front face is not a true involute curve. We will examine it to determine the nature and amount of this error in form. The correct cutting edge has the equation

$$\theta = \sqrt{\left(\frac{r}{a}\right)^2 - 1} - \tan^{-1}\sqrt{\left(\frac{r}{a}\right)^2 - 1} \quad \text{(see Eq. (14))}$$

We will calculate three points on this curve, as follows, when

$$\begin{aligned} r &= \text{outside radius } E \\ r &= \text{pitch radius } R \\ r &= \text{radius of base circle } a_1 \end{aligned}$$

Taking, as an example, a 4-d.p., 20-deg. stub-tooth cutter of 3-in. pitch diameter, we have the following:
When,

$$r = a_1 = R \cos 20 \text{ deg.} = 1.4095 \text{ in.}$$

$$\theta = \sqrt{\left(\frac{a_1}{a_1}\right)^2 - 1} - \tan^{-1}\sqrt{\left(\frac{a_1}{a_1}\right)^2 - 1} = 0$$

When

$$r = R = 1.500 \text{ in.}$$

$$\theta = \tan 20 \text{ deg.} - \text{arc } 20 \text{ deg.} = 0.01490$$

When

$$r = E = 1.750 \text{ in.}$$

$$\left(\frac{r}{a_1}\right)^2 - 1 = \left(\frac{1.750}{1.4095}\right)^2 - 1 = 0.54142$$

$$\sqrt{\left(\frac{E}{a_1}\right)^2 - 1} = 0.73581$$

$$\theta = 0.73581 - \tan^{-1}\ 0.73581 = 0.73581 - 0.63435 = 0.10146$$

The actual cutting edge, however, has the equation,

$$\theta = -Gr + \sqrt{\left(\frac{r}{a_2}\right)^2 - 1} - \tan^{-1} \sqrt{\left(\frac{r}{a_2}\right)^2 - 1}$$

Thus, when $r = a_1 = 1.4095$ in.

$$\theta = -Ga_1 + \sqrt{\left(\frac{a_1}{a_2}\right)^2 - 1} - \tan^{-1} \sqrt{\left(\frac{a_1}{a_2}\right)^2 - 1}$$

$$a_2 = R \cos \alpha' = 1.4078 \text{ in.}$$

$$\sqrt{\left(\frac{a_1}{a_2}\right)^2 - 1} = 0.05026.$$

$$G = \frac{2\pi}{K} \tan \gamma$$

$$\gamma = 5 \text{ deg.}$$

$$\frac{2\pi R}{K} = \tan \Sigma = \tan 7 \text{ deg.} \tan 20 \text{ deg.}$$

whence,

$$K = \frac{2\pi R}{\tan 7 \text{ deg.} \tan 20 \text{ deg.}} = 210.89268 \text{ in.}$$

$$G = \frac{2\pi \tan \gamma}{210.89268} = 0.00261$$

$$\theta = -Ga_1 + 0.05026 - \tan^{-1} 0.05026 = -0.00360$$

when

$$r = R = 1.500 \text{ in.}$$

$$\theta = -GR + \sqrt{\left(\frac{R}{a_2}\right)^2 - 1} - \tan^{-1} \sqrt{\left(\frac{R}{a_2}\right)^2 - 1} = 0.01146$$

When

$$r = E = 1.750$$

$$\theta = -GE + \sqrt{\left(\frac{E}{a_2}\right)^2 - 1} - \tan^{-1} \sqrt{\left(\frac{E}{a_2}\right)^2 - 1} = 0.09782$$

Tabulating the foregoing values, we have

Value of r	θ for involute profile	θ for actual profile
a_1	0	−0.00360
R	+0.01490	+0.01146
E	+0.10146	+0.09782

To obtain the angular error of the actual profile, we must use the same polar coordinate system for both curves. Considering the point at R of both curves as being identical, we must move the points on the actual profile ahead the circular amount of $0.01490 - 0.01146$, which equals 0.00344. This gives the following tabulation:

Value of r	θ for involute profile	θ for actual profile	Angular error in profile
a_1	0	-0.00016	-0.00016
R	$+0.01490$	$+0.01490$	0
E	$+0.10146$	$+0.10126$	-0.00020

In order to measure the error as a length on the circumference of a circle, we must multiply the foregoing angular errors by the radius of the corresponding circle. This gives the following errors for the profile of the actual cutting edge:

Value of r	Error in cutting profile, inches
a_1 (1.4095)	-0.00021
R (1.5000)	0
E (1.7500)	-0.00032

The deviation of the profile from the true involute is -0.00032 in. at the outside diameter, zero at the pitch diameter, and -0.00021 in. at the base circle.

The actual cutting profile is plotted against a true involute profile in Fig. 174. The error is small and is less than the other probable errors which develop in the mechanical processes of making and using these cutters. The actual profile is less curved than the theoretically correct one. The cutter, therefore, removes somewhat more material at the tip and root of the generated gear than it ought to do. An error in this direction is preferable to one in the opposite direction, which would leave too much metal on the gear teeth at these points and tend to develop interference.

Great care should be exercised when resharpening these cutters. A change in the angle of the conical-front face alters the profile of the cutting edge so that it would generate gears of a slightly different normal pitch, or pressure angle at a fixed-

center distance, than it would when this angle is correct. This conical grinding must be concentric with the cutting edges, else these cutting edges will become eccentric. Due to the relieved surfaces of the cutter, carelessness in resharpening will impair the accuracy of the cutting edges.

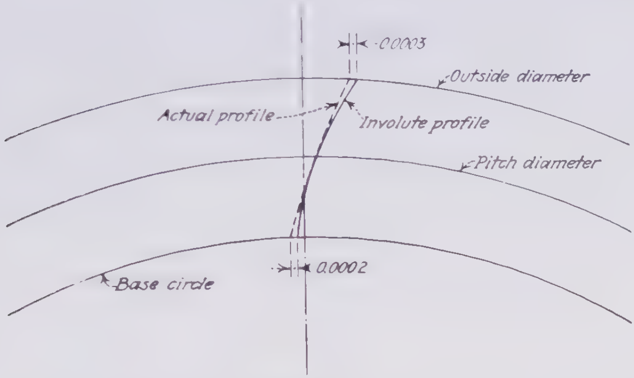


Fig. 174.—Error on cutting edge of a pinion-shaped cutter.

It is of interest to determine the effect of grinding these cutters to a different cone angle at the front face. We have the following equation for the tangent to the profile of the relieved surfaces at the pitch line:

$$\tan \alpha' = \tan \alpha + \tan \gamma \tan \Sigma$$

Transposing, we have

$$\tan \alpha = \tan \alpha' - \tan \gamma \tan \Sigma \tag{187}$$

Substituting different values for the cone angle γ will give us the altered tangent to the profile of the cutting edge at the pitch line. Thus, for a 3-in. pitch-diameter cutter with a nominal pressure angle of 20 deg., we have previously obtained the following values, which will be constant for these calculations:

$$\tan \alpha' = 0.36788$$
$$\tan \Sigma = 0.04469$$

Solving the foregoing equation for values of γ from 0 to 10 deg., we get the following values for the angle of the tangent to the profile of the cutting edge at the pitch line, or pressure angle. This tabulation also includes the normal pitch for 1 d.p.

Front-cone angle	Pressure angle	Normal pitch, 1 d.p., inches
0° 0′	20° 11′ 51″	2.94842
0° 30′	20° 10′ 40″	2.94878
1° 0′	20° 9′ 30″	2.94915
1° 30′	20° 8′ 19″	2.94952
2° 0′	20° 7′ 8″	2.94989
2° 30′	20° 5′ 57″	2.95027
3° 0′	20° 4′ 46″	2.95064
3° 30′	20° 3′ 34″	2.95101
4° 0′	20° 2′ 22″	2.95138
4° 30′	20° 1′ 11″	2.95175
5° 0′	20° 0′ 0″	2.95212
5° 30′	19° 58′ 49″	2.95249
6° 0′	19° 57′ 38″	2.95286
6° 30′	19° 56′ 26″	2.95323
7° 0′	19° 55′ 13″	2.95361
7° 30′	19° 54′ 1″	2.95399
8° 0′	19° 52′ 48″	2.95437
8° 30′	19° 51′ 35″	2.95475
9° 0′	19° 50′ 22″	2.95513
9° 30′	19° 49′ 9″	2.95551
10° 0′	19° 47′ 56″	2.95589

It will be seen from the foregoing tabulation that a change of 1 deg. from the normal front-cone angle of 5 deg. will change the normal pitch of a 1-d.p. cutter, 0.000074 in. For a 6-d.p. cutter, the change would be one-sixth of this amount, or 0.000012 in. The normal pitch increases with an increase in this cone angle and decreases with a decrease of the cone angle. Advantage of this characteristic of pinion-shaped cutters can sometimes be taken, either to correct for small errors in the cutters or to maintain a definite normal pitch relationship between mating gears.

The cutting action of a pinion-shaped cutter is very similar to that of an ordinary shaping or planing tool, except that the size of the chip removed is not constant at every stroke. As with hobbing, when the first roughing cut is made, most of the cutting takes place on the entering side of the gear blank. By the time that a tooth space of the gear has reached the common center line of gear blank and cutter, most of the metal has been removed.

Again, as with hobbing, the cutting edge at the outside diameter of the cutter is not only relatively short, but on roughing cuts it also removes about one-half of the metal.

Effect of Irregular Loads.—The cutting edges at the sides of the cutter do not cut continuously. First one side will be cutting, then both sides together, then the opposite side only, etc. With this process, as with hobbing, the tendency of these side-cutting edges of the cutter is to drive the gear blank ahead at one instant and to hold it back at another instant. This combination of irregular loads, both in amount and direction, makes it necessary to take individual finishing cuts on the two sides of the gear teeth in order to reproduce the maximum of the accuracy of the gear-shaping machine and cutter.

The tooth-profile surfaces of gears shaped with pinion-shaped cutters are composed of a series of hollows and ridges extending across the face of the gear. The feed on finish cuts must be fine enough so that these ridges will be readily crushed down when operated under load. These continuous ridges resist this crushing more than do the isolated peaks left by hobbing.

These ridges are not evenly spaced along the tooth profile but are very much closer together below the pitch line than they are above it. The feed should therefore be selected to give a sufficiently fine feed on the outer part of the tooth profile.

The most effective use of this process of shaping gear teeth with pinion-shaped cutters is on gears of relatively small diameters and fine pitches. This process is extensively used in the production of gears for the automotive industry, where the sizes are relatively small and the pitches are relatively fine. The limitations of the cutter, as pointed out before, preclude its use on large gears of coarse pitch.

This process has the advantage of being able to cut gear teeth that are close to a shoulder. Thus, when space and weight are limited, several gears of different diameters can be made integral with a narrow recess between them, and the teeth can be generated by this method.

This process can also be used in the production of sprockets for chain drives, small cams, ratchets, etc., by providing a suitable cutter which is conjugate to the form to be produced.

SHAPING GEAR TEETH WITH A RACK-SHAPED CUTTER

Another method of shaping gear teeth consists of the use of a rack-shaped cutter with a special shaping or planing machine. The Sunderland gear-shaping machine which was developed in England is one machine of this type. The Maag gear shaper,

which was developed in Switzerland, is a further development and refinement of the Sunderland machine. In Fig. 175, one of the smaller Maag gear shapers is shown.

These machines have a vertical work spindle, while the rack-shaped cutter is carried in a reciprocating ram. In general, it resembles the Fellows gear shaper, with the substitution of a rack-shaped cutter for the pinion-shaped one, and a change in the

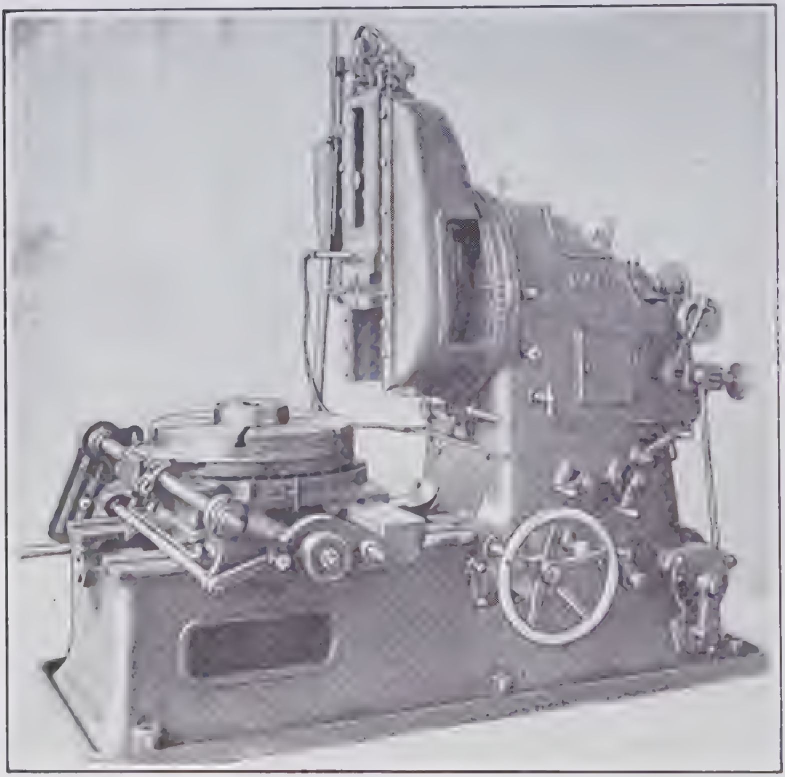


Fig. 175.—Small-size Maag rack-type gear shaper.

motion of the work table from a rotary one to a combined roll and linear advance. In operation, the gear blank is revolved and advanced the corresponding distance along the face of the rack-shaped cutter between each cutting stroke. When the blank has advanced a distance equal to the circular pitch, it returns to the starting position without any rotary motion, then the same cycle of operations is repeated until all the teeth in the

blank have been completed. Thus, the successive teeth in the gear are generated by the same two or three teeth in the rack-shaped cutter. Figure 176 shows one of these rack-shaped cutters meshing with a partially finished gear blank. Here, also, as with hobbing and shaping with pinion-shaped cutters, the accuracy of the product depends upon three primary factors: the accuracy of the machine; the accuracy of the cutter; and the carefulness of the operator in mounting the gear blanks, resharpening the cutters, etc.

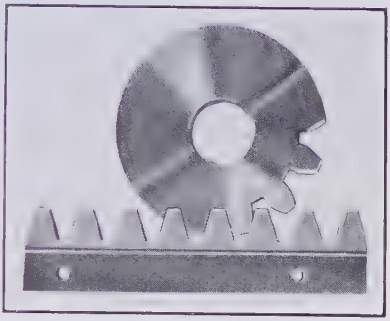


FIG. 176.—Rack-shaped cutter meshing with partially finished blank.

In many respects, the problems of securing and maintaining the required accuracy of the machines are very similar to those of hobbing machines and gear shapers using pinion-shaped cutters. Inaccuracies in the functional parts of the machine will be reproduced in the product. Also, the construction must be sufficiently rigid to prevent excessive deflection or vibration when the machine is in operation. In one important respect, however, the conditions here are quite different from those present in hobbing machines and shaping machines using pinion-shaped cutters. In the rack-cutter shaping machine, all of the generating mechanism is concentrated in the table of the machine and is not interconnected with any movement of the cutter, as is necessarily the case with any revolving generating tool. This generating motion consists of an index worm and wheel and a pitch screw connected by change gears to enable any required relative rotary and linear movements to be obtained. In Fig. 177, a diagram of this mechanism in its elementary form is shown. This method of shaping gear teeth has the same advantage as have all other generating processes, in that one cutter will produce mating gears of any tooth numbers. It is interesting to note that, while a pinion-shaped cutter is mathematically a multiple-threaded hob with an infinite lead, the rack-shaped cutter is both a section of a hob of infinite diameter and a section of a pinion-shaped cutter of infinite diameter. This process is used to produce gears of all sizes. Figure 178 shows a machine

with a capacity to cut gears up to 40 ft. in diameter. All of the effective cutting edges of a rack-shaped cutter that is used to produce involute gears are straight lines. These cutting edges are formed by the intersection of the plane-relieved surfaces with the tool-cutting face, which is also a plane. The mathematical analysis of this type of cutter is, therefore, extremely simple.

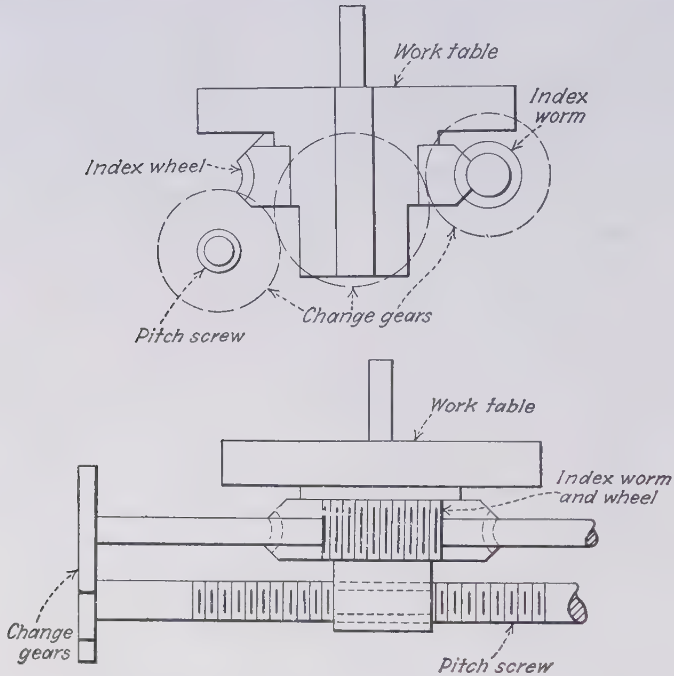


FIG. 177.—Elements of the generating mechanism of a rack-type gear shaper.

Referring to Fig. 179, which is a drawing of one of these cutters, let

α = projected angle of cutting edge on cutting plane, or pressure angle of basic-rack form

α' = half included angle between relieved-side surfaces of cutter teeth

γ = angle of cutting face of cutter in cutting position

ϵ = angle of relief of tip of cutter tooth

β = angle of relief of sides of cutter teeth

$$\tan \alpha' = \frac{\tan \alpha \cos \gamma}{\cos (\gamma + \epsilon)} \tag{188}$$

$$\tan \beta = \tan \alpha \tan \epsilon \tag{189}$$

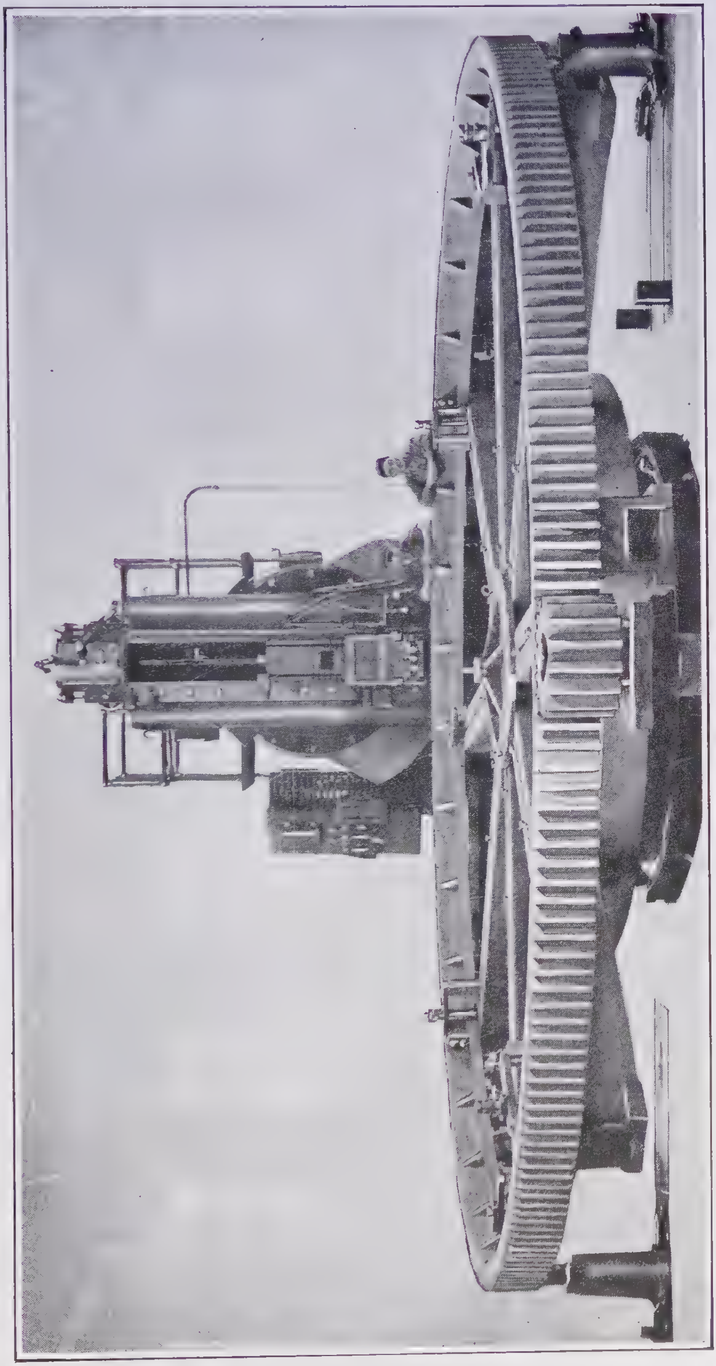


FIG. 178.—Large-size rack-type gear shaper.

No mathematical errors exist in the profile of the cutting edges of these cutters. As has been pointed out before, the introduction of relief and rake on hobs and pinion-shaped cutters also introduces certain mathematical errors. These errors are usually small and are in the most advantageous direction, that is, they ease off or modify the ends of the involute profiles of the generated gears and, hence, tend to prevent edge contact. They also tend to minimize the effects of other errors which may be present. When such modification is desirable, it can also be

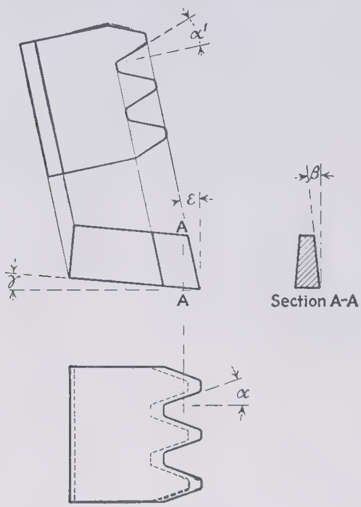


Fig. 179.—Design of a rack-shaped gear cutter.

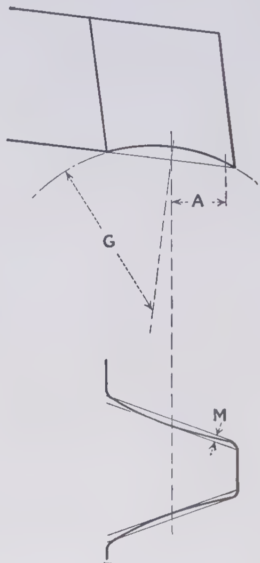


Fig. 180.—Modified rack-shaped gear cutter.

introduced on rack-shaped cutters by grinding the front-cutting edge to the form of a hollow cylinder instead of flat. In Fig. 180, this is illustrated.

Referring to Fig. 180, let

M = required amount of modification or departure from a straight-line profile, inches

A = addendum or one-half the working depth of the cutter tooth, inches

G = radius to grind cutting face of cutter, inches

β = angle of relief of sides of cutter teeth

$$G = \frac{M^2 + A^2 \tan^2 \beta}{2M \tan \beta} \tag{190}$$

As a definite example, we will take a 4-d.p. cutter for 20-deg., full-depth tooth gears with a tip relief of 7 deg. and a rake angle of 5 deg. on the front-cutting face, and calculate for a modification of 0.0002 in., in which case,

$$\alpha = 20 \text{ deg.}$$
$$\gamma = 5 \text{ deg.}$$
$$\beta = 7 \text{ deg.}$$
$$M = 0.0002 \text{ in.}$$
$$A = 0.2500 \text{ in.}$$
$$\tan \beta = 0.36397 \times 0.12278 = 0.04469$$
$$G = \frac{(0.0002)^2 + (0.250 \times 0.04469)^2}{2 \times 0.0002 \times 0.04469} = 6.98 \text{ in.}$$

In this example, if the front face of the cutter is ground to a radius of about 7 in., the profile of the side-cutting edges would depart very closely to 0.0002 in. from a straight line.

The cutting action of a rack-shaped cutter is almost identical to that of a pinion-shaped cutter, except that the cutting edges remain in contact with the final tooth profile over a slightly longer arc of action. The cutting load here is also irregular both in direction and amount, so that, in order to reproduce the maximum accuracy of the machine and cutter, it is necessary to finish the opposite sides of the gear teeth by individual finishing operations.

The surface of the tooth profiles of gears produced by rack-shaped cutters is a series of flat surfaces with their intersections or ridges running across the face of the gear. This surface is almost identical to that produced by pinion-shaped cutters. If the feed used to cut the teeth is sufficiently fine, these ridges will break down and burnish into a smooth, continuous surface, when the gears are operated under load, either in service or as a smoothing or burnishing operation.

The limitations of gear-generating methods are established primarily by the limitations of the cutting tools. The rack-shaped cutter holds the unique position of being a limiting case of both the hob and the pinion-shaped cutter and is not affected by the theoretical limitations of either of these two types of tools. Thus, the only limitations of this process are those of machine construction.

The following comparisons are not made as criticisms of any of the methods but are based on the possibilities of these methods

rather than on their actual status at the present time. These comparisons are made solely for the purpose of bringing out as clearly as possible the salient features of each of them.

On small and medium-sized gears, the hobbing process seems to offer the greatest possibilities of production. The continuous rotary motions employed in these machines are usually ideal for speed. On this same range of size of gears, the pinion-shaped cutter method and the rack-shaped cutter method are nearly equal, with the pinion-shaped cutter method slightly in the lead. Both use a reciprocating cutting stroke, but the pinion-shaped cutter revolves continuously, while the generating action of the rack-shaped cutter is intermittent. The possible speed of reciprocation is the same on both machines. The rack-shaped cutter, however, requires slightly fewer cuts to obtain the same quality of surface than does the pinion-shaped cutter, but this advantage is not quite enough to overcome the handicap of stopping the generating action at the end of each tooth cycle.

On large gears of coarse pitch, the conditions are changed. The hob, of necessity, must be made of larger diameter, so that its rate of revolving is materially reduced while its feed cannot be materially increased. The pinion-shaped cutter method cannot be used here without the development of very much larger cutters and machines than exist at present. The rack-shaped cutter method suffers no reduction in cutting speed because of cutter limitations, while its percentage of idle time due to its intermittent generating action is materially reduced, so that here it proves faster in production.

The machines used to generate gears as now made are about equally accurate in their operation. The characteristics of the generating mechanisms are such, however, that much more care and effort is required to attain this result on one type of machine than on another. In fact, this same condition holds true on different designs of the same type of machine.

The generating mechanism for the rack-shaped cutter method is the simplest, as it consists of but three main elements, namely, the index worm and wheel, the pitch screw, and the change gears. The pitch screw, however, is subject to local wear and needs periodical attention, just as does the lead screw of a lathe, if the highest order of accuracy is to be maintained.

The generating mechanism of the hobbing machine comes next in order in relation to simplicity, as it consists of an index worm

and wheel, hob spindle and drive, and change gears. Usually, the gearing between the hob spindle and index worm goes around two or three corners because of the necessity of providing angular adjustment to the hob spindle. This complicates the drive and makes it that much more difficult to obtain the required degree of accuracy. None of the important functional parts is subject to local wear, however, but the index worm and wheel are operating many times oftener here than on the other two types of machines.

The generating mechanism of the pinion-shaped cutter machine is the most complex, consisting of the work index worm and wheel, cutter index worm and wheel, gear drive between the two index mechanisms, and change gears. Here the drive between the two index mechanisms also goes around two corners. None of the important functional parts of these machines is subject to local wear, however.

As with the machines themselves, the different types of generating tools as made are of about equal accuracy. Their characteristics are such, however, that more care and effort is required on one type than on another in order to attain this result.

The rack-shaped cutter is the easiest to make accurately. All surfaces, being planes, are readily made and measured accurately. Furthermore, there is no mathematical error introduced because of relief. In addition, if a slight modification or departure from true involute form is deemed advisable to compensate for other errors, this departure can be made and definitely controlled within specified limits by the grinding of the front-cutting face of the cutter.

Although the tooth form on a pinion-shaped cutter is more sensitive than that on a hob, in many respects it is easier to make it accurately. For one thing, when grinding the relieved surfaces, there is room to use a grinding wheel of any desired size, while on a hob with its succession of cutting flutes, the size of the grinding wheel is limited by the small amount of clearance between the wheel and the succeeding cutting edge. Also, the relieved surfaces of pinion-shaped cutters used to produce spur gears are symmetrical, so that variations in the front-cone angle may be used when desired to compensate for other factors, while, on a hob, the relieved surfaces on opposite sides of the teeth are not symmetrical. On the other hand, cumulative errors in spacing on pinion-shaped cutters are particularly objectionable, because the successive teeth in the gear blank are generated by successive

teeth on the cutter, so that under certain conditions of relative tooth numbers of the gear and cutter, all of this cumulation may exist between the first and last tooth space cut. With rack-shaped cutters and single-threaded hobs, each tooth space in the gear blank is generated by the same two or three teeth on the cutters, so that cumulative spacing errors on these cutters have but little effect on the profile of the gear tooth produced and no effect at all on the spacing of the teeth of the generated gear. With multiple-threaded hobs, these conditions begin to approach those of the pinion-shaped cutter.

A comparison of the cutting action of the different generating tools for gears resolves itself mainly into a discussion of the relative merits of milling and shaping. At the present time, this is still an open question between these two processes when they are applied to the machining of other types of surfaces than gear-tooth profiles. It seems to be largely a matter of opinion. Furthermore, the physical characteristics of some materials are such that they will take a smoother milling cut than a shaping cut, while other materials will take a smoother shaping cut than a milling one. Both shaping methods have the advantage, however, of being able to cut gear teeth close to shoulders.

It is apparent, then, that no one method has all the advantages. Specific requirements and conditions determine which is best to use.

CHAPTER XII

GRINDING OF GEAR TEETH

The introduction of each new or improved method of finishing the profiles of gear teeth seems to be hailed as the means of eliminating all future gear troubles. When this hope is not fulfilled, a counterreaction takes place. This seems to be particularly true of the grinding processes for gear teeth. The fault, however, is not entirely that of the process. Gear-tooth grinding is a comparatively recent development and unquestionably needs still further development before it will become a general production process. The larger part of this further improvement, however, is in the development of the skill and technique of the operators in charge of such grinding equipment rather than in radical changes in the design of the tooth-grinding equipment. This skill and technique are not necessarily required of each individual operator, but someone in immediate charge of such grinding must have it to secure satisfactory and consistent results.

The grinding of gear teeth is an entirely different problem from that of any of the other production methods. For example, with milling, hobbing, and shaping, there are three primary factors that control the accuracy of the product, namely, the machine, the cutting tools, and the skill of the operator. In all of these processes, the larger part of the real gear skill and technique goes into the production of the cutting tools. The larger part of the recent improvements in all of these processes has been improvements in the accuracy of these cutting tools. The accuracy of the product produced by the gear-tooth grinding process, however, depends upon but two primary factors, namely, the accuracy of the grinding machine and the skill and technique of its operator. There is no small-tool manufacturer here to take the brunt of building into the cutting tools the skill and technique required in the production of accurate gear teeth.

In addition, the general trend of production methods today is toward the development of manufacturing processes that require

a minimum of skill on the part of the operators. Under these circumstances, it should be evident why the first attempts to grind gears on a production basis generally have not been entirely satisfactory.

The grinding of gear teeth is a valuable and necessary process. Its use will become more and more general as its value proves itself. But this can be accomplished only by facing the fact frankly that this is one production process where skill and technique cannot be eliminated from the production floor. Possibly, the final solution will be the growth of more gear-grinding plants that will specialize on this process only.

Some method of finishing the tooth profiles of hardened-steel gears is very necessary. Wherever maximum strength with minimum weight is required, hardened-steel parts are used. This has led to the use of hardened-steel gears in automobiles, airplanes, machine tools, etc. Gear-tooth profiles are very sensitive surfaces, and even small distortions cause much trouble. Such errors not only cause noisy operation but also reduce the load-carrying capacity of the gear teeth. The additional load caused by errors in gear-tooth profiles seems to be somewhat proportional to the extent of the error. Any increase in the accuracy of the tooth profiles of hardened-steel gears has, therefore, a very real value. The higher the pitch-line velocities of such gears, the greater this value will be. Today, hardened and ground spur gears are being successfully operated at pitch-line velocities of 16,000 ft. per minute. Practically all ground gears as finished have a very high-pitched tone when operated under load at relatively high speeds, quite different from the tone of cut gears. This tone is caused by the fine grain of the ground surfaces, which are never absolutely smooth. If these grinding marks are not too deep, this surface soon wears or polishes to a very smooth one, and the amount of this sound is reduced until it practically disappears. In many cases, this difference in the character of the sound of ground gears has been responsible for their condemnation, primarily because it has been different from the usual sound of cut gears. In order to judge the smoothness of operation of ground gears by ear, it is necessary to listen through this high-pitched tone for the other characteristic sounds caused by faulty tooth profiles, spacing, and the like.

One troublesome source of error on almost all gear-tooth grinding machines is the wear on the grinding wheel. This introduces

the problem of selecting the most suitable grade and grain for such grinding wheels. If the wheel is hard, it will hold its form longer but cannot grind so fast, because of its tendency to glaze. If the wheel is soft, it will cut more freely but will lose its form sooner. This holds true on all types of gear-tooth grinding machines. The final answer here, as in practically all other production problems, is a compromise between these conflicting elements. A suitable grinding wheel must be selected to meet the particular conditions that exist with each different type of material to be ground.

If the teeth of the gear are ground successively in order, wear on the grinding wheel will introduce a maximum cumulative error between the first and last tooth space ground. This maximum cumulative error may be greatly reduced by using a suitable multiple-tooth indexing arrangement. Thus, instead of indexing one tooth interval at a time, a multiple-tooth indexing interval may be selected such that the first and last tooth ground are never adjacent. For example, if a gear with 20 teeth is to be ground, and the gear is indexed nine tooth intervals each time, the tooth spaces would be ground in the following order: 1, 10, 19, 8, 17, 6, 15, 4, 13, 2, 11, 20, 9, 18, 7, 16, 5, 14, 3, 12, and then the same cycle would be repeated again if two cuts were taken. In this way, the effects of wear on the grinding wheel are distributed around the circumference of the gear and do not result in a large cumulative error between adjacent teeth.

Gear-tooth grinding machines may be divided into two general classes: first, those that use a formed wheel; and second, those that use a grinding wheel, the form of which represents the basic rack of the gear-tooth system, in combination with a molding or generating action.

In principle, the formed-wheel grinding process is very similar to the form-milling process, with the substitution of a formed grinding wheel for the milling cutter. This process, developed by the Gear Grinding Machine Company, of Detroit, Michigan, was among the first gear grinding processes successfully used in production. In Fig. 181, a machine of this type, of the size used to grind gears for automobile transmissions, is shown.

In operation, the grinding wheel is trued in position by a truing device located at the end of the work table. The operation of this truing device is controlled by accurate templates of enlarged form. The grinding wheel is finish trimmed immedi-

ately before the finish grinding of each set-up of the gears in the machine, to insure accuracy of the final cut.

The truing device must be accurately located in relation to the position of the gear blank, in order that the tooth profiles be correctly positioned. When once this position has been secured, no further change is required so long as similar gears are to be produced.

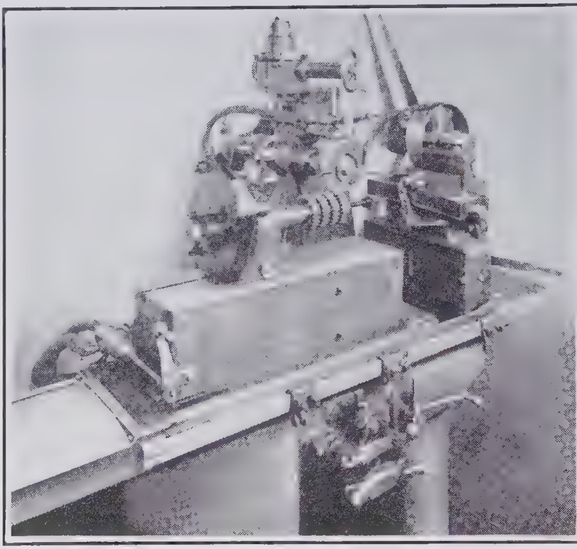


Fig. 181.—Small form-wheel type of production gear grinder.

The functional elements of this machine have been reduced to a minimum, consisting primarily of an indexing mechanism and a formed wheel with suitable means for maintaining the form on the wheel. Gear-tooth profiles of any desired form can be produced by providing suitable templates to control the operation of the truing mechanism. A larger machine of this type is shown in Fig. 182. This machine has a capacity of pitch diameters up to about 50 in. and face widths up to about 21 in.

There are several designs of gear-tooth grinding machines now on the market that employ a generating or molding action with a flat-faced grinding wheel. All of them employ the same fundamental principles. The earliest successful and commercial machine of this type was one developed by the Fellows Gear

Shaper Company to grind the relieved surfaces of their pinion-shaped cutters.

Practically all machines of this type employ a flat-faced grinding wheel tipped at an angle to represent the side of the basic-rack tooth of the gear-tooth system. In order to generate the involute gear-tooth profile, the gear blank is rolled past this grinding wheel with the same rolling motion as when an accurate gear is meshing with its basic rack. An index plate is usually

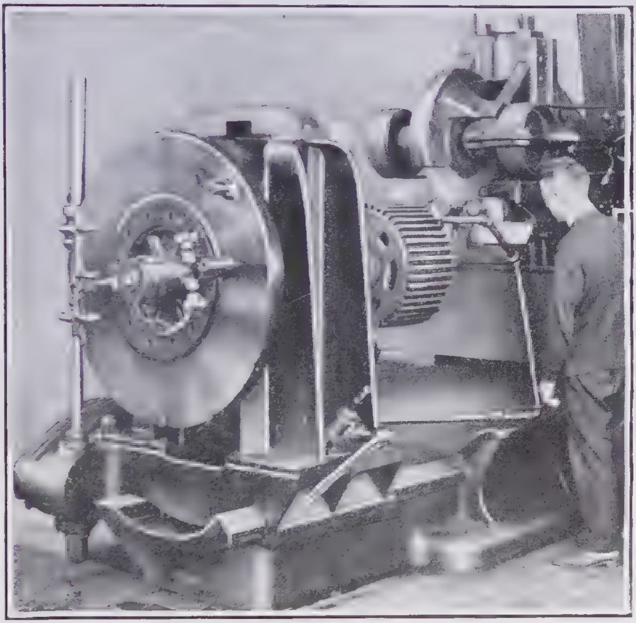


Fig. 182.—Large-size gear grinder of the form-wheel type.

employed to index the blank, so as to grind the successive tooth profiles. Here, as with formed-wheel grinding, a multiple-tooth indexing arrangement could be used to advantage to prevent the first and last tooth ground from being adjacent.

One design of this type of gear-tooth grinding machine is the Lees-Bradner machine, illustrated in Fig. 183. This machine uses a single grinding wheel and grinds one side of the tooth profiles at each set-up.

The rolling motion on this machine is controlled by a pitch disk of the same diameter as the pitch diameter of the gear to

be ground. To this pitch disk are attached two, thin, flexible-steel tapes, as shown in the diagram in Fig. 184, which are attached at their opposite ends to the bed of the machine and

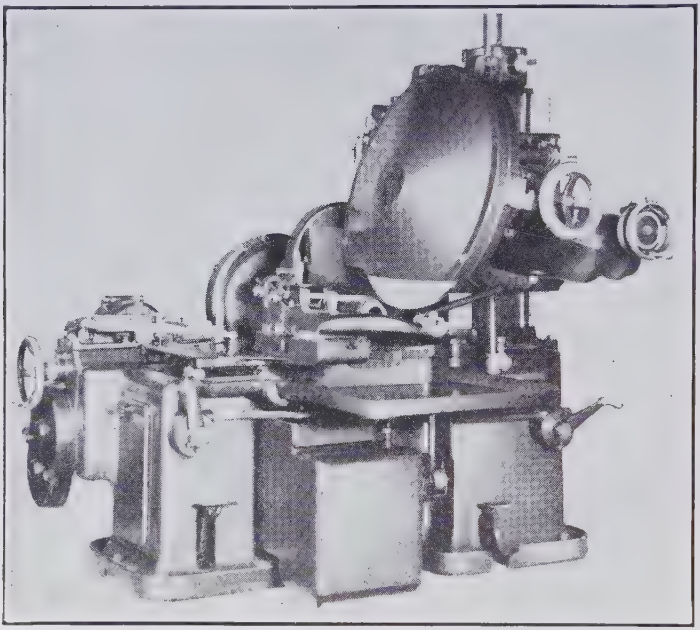


FIG. 183.—Lees-Bradner generating type of grinder.

stretched tightly, thus causing the work spindle to rotate without lost motion or slippage as the carriage of the machine is moved from side to side under the grinding wheel.

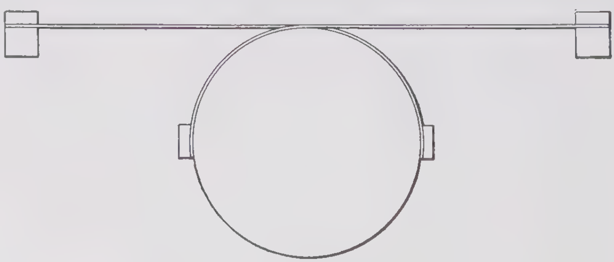


FIG. 184.—Pitch disk of the Lees-Bradner gear grinder.

As noted before, if an involute gear has perfect tooth profiles perfectly spaced, no modification of the involute tooth profile is necessary. As errors of even very small amounts often cause

noise in operation, particularly if edge contact exists at the beginning of mesh, a very slight modification often proves advisable to compensate for other small errors. On grinding machines of this type, such modification to avoid edge contact at the beginning of mesh may be obtained in one of two ways: First, the normal pitch of the driving gear may be made slightly larger than that of the driven gear, by changing the angle of the grinding wheel. If this angle (pressure angle) is made smaller, the normal pitch will be increased. If it is made larger, the normal pitch will be decreased. The second method is to set the pitch disk to which the tapes are attached slightly eccentric or off center with the work spindle in a vertical direction. The amount of such modification required depends primarily upon the accuracy of the other elements of the machine and must be established by trial. Such modifications, furthermore, should always be kept to an extremely small amount. Gears whose other elements are so inaccurate that they require excessive modification to compensate for these errors can never be fully satisfactory.

The flat-faced grinding wheels used on these machines are trued by a diamond dressing tool mounted in an arm that swings in a plane perpendicular to the axis of the grinding wheel. This diamond truing device is set in a fixed position, so that the grinding wheel is fed into it for redressing, thus maintaining the grinding face of the wheel in a fixed position.

An earlier design of this same type of grinding machine was developed by the Brown-Lipe Company, of Syracuse, New York. This machine used two grinding wheels, thus finishing both sides of the gear-tooth profile in the same operation. The work spindle is also in a vertical position in front of the grinding wheels. The grinding wheels on these machines are very large, about 5 ft. in diameter, and are constructed of small segments fastened to a large cast-iron disk. The principle of operation is the same as for the previous machine described.

The Fellows Gear Shaper Company has also developed a gear-tooth grinding machine for automotive gears. These machines have but a single grinding wheel, but they are so designed that two or more units may be mounted on a single base, some to grind one side of the tooth profiles while the others grind the opposite tooth profiles. Instead of using a pitch disk and tapes to control the rolling motion of the work spindle, these machines

have cams of involute form acting against straight-edges that have angular adjustment. These straight-edges represent the side of a tooth of a basic rack.

Another machine of this same general type is one developed by the National Tool Company of Cleveland, Ohio. In Fig. 185, the cam type of control is clearly shown.

All of the foregoing generating types of gear-tooth grinding machines have been designed to finish gears of relatively small diameters and narrow faces, such as are used in the change-gear transmissions of automobiles. Another design of this same general type of machine, designed for larger diameters and wider faces, is the Maag gear-grinding machine, illustrated in Fig. 186. This machine was developed in Switzerland, uses two grinding wheels to grind both sides of the tooth profiles at once, and has many interesting and unusual features.

The grinding wheels on this machine are saucer shaped, and the grinding is done by the edge of the wheel instead of on a flat face. This edge tends to true itself as it wears. The work table moves under the wheel in the direction of the axis of the work spindle, thus causing the edges of the grinding wheels to cover planes that represent the sides of the basic-rack teeth. To compensate for the wear on the grinding edges of the wheels, which is appreciably greater than on a flat-faced wheel, a very effective positioning device is employed. At the upper edge of each wheel is mounted a pivoted lever carrying a diamond with a flat, polished face, as shown in Fig. 187. About every 6 sec., this pivoted lever is allowed to swing until the flat face of the diamond touches the edge of the grinding wheel. If this edge has worn, the lever swings beyond its zero position and makes an electrical contact which causes an electromagnet to act on a lever carrying a feed pawl, thus feeding the ratchet wheel attached to a differential screw one notch. This ratchet must be moved about 20 notches to feed the grinding wheel into the work a distance of .001 inch. If the wheel has not worn, no contact is made, and the wheel remains in the same position. This positioning mechanism works continuously all the time while a gear is being ground, and the position of the wheel is corrected in the middle of a cut.

This device has proved to be remarkably sensitive and reliable. When the wheel has worn very slightly, the contact made is not sufficient to cause the magnet to move the pawl the full distance of one notch. In addition, a push button is provided which

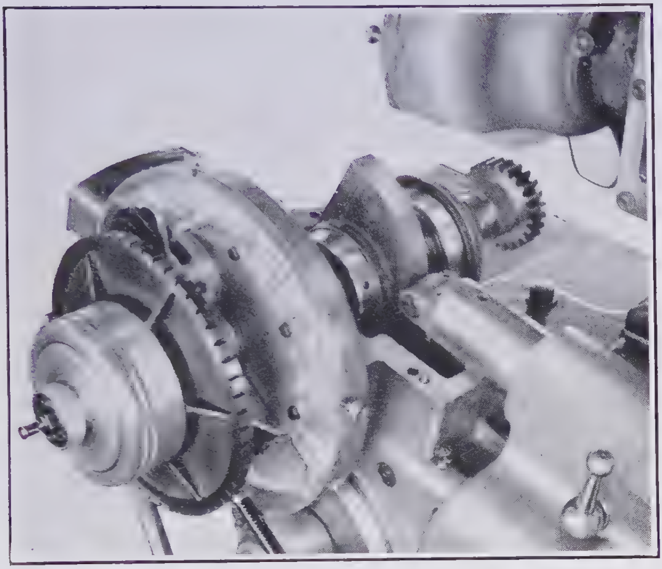


FIG. 185.—Cam control of profile on National gear grinder.

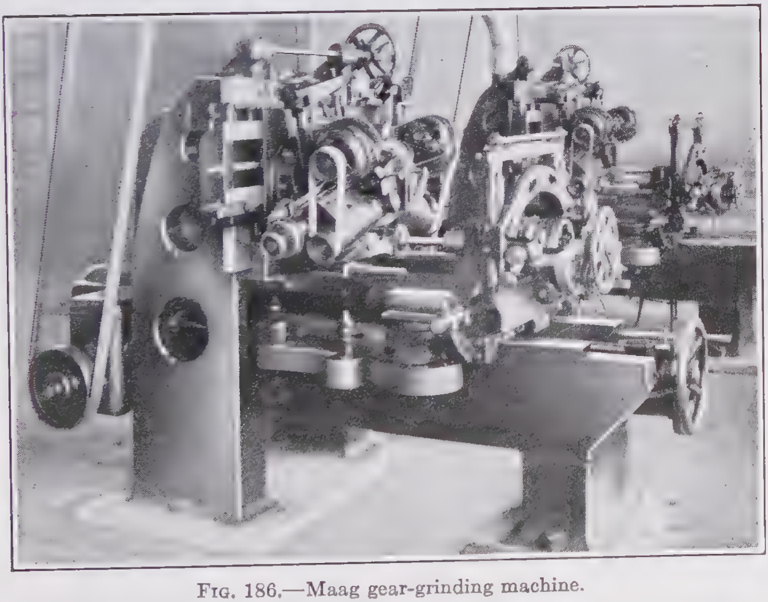


FIG. 186.—Maag gear-grinding machine.

operates this positioning device independently of the automatically timed device, so that the position of the wheel may be set at random away from the work, and this button may be pushed successively until no further action on the ratchet takes place. When a wheel is repeatedly displaced and repositioned in this manner, this device will repeat its setting to within about two notches on the ratchet wheel. The pressure on the lever carrying the flat-faced diamond must be extremely light, otherwise the flat, polished face of the diamond will be destroyed.

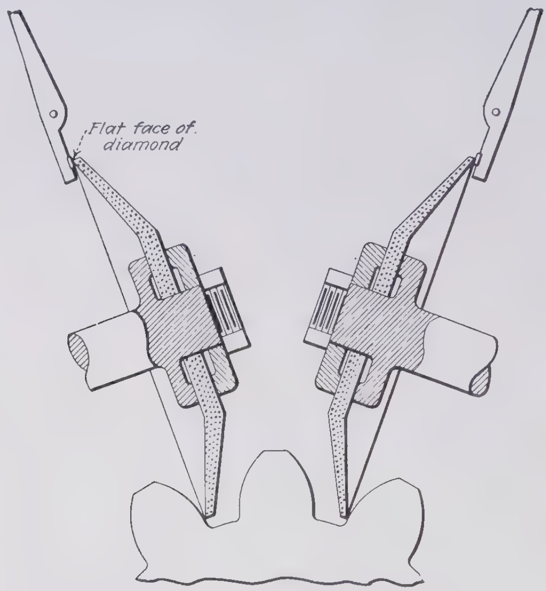


Fig. 187.—Wear-compensating device on the Maag grinder.

Some diamonds seem to be harder than others. It is interesting to note, however, that some of these diamonds have been used continuously for over 4 years without showing under the microscope any signs of cutting or scratching by the grinding wheels.

Another interesting feature of these machines is the pitch-disk mechanism. It will be noted, in Fig. 186, that the pitch disk is nearly double the pitch diameter of the gear being ground. The work spindle is mounted on a cross-slide, while the ends of the tapes are attached to a second cross-slide, so that, by means of a compensating bar, the roll on this larger pitch disk may be made correct for any pitch diameter within the range of this adjustable

compensating mechanism. This feature has two advantages: first, that a single pitch disk may be used to cover a considerable range of pitch diameters; and second, that the pitch disk is always larger than the pitch diameter of the gear to be ground, which is of particular advantage on gears of small diameter.

These machines have a capacity up to 16 in. in pitch diameter and faces up to about 12 in. Figure 188 shows a larger gear of wide face in position on one of these machines. A larger model of this same machine will grind gears with pitch diameters up to about 36 in.

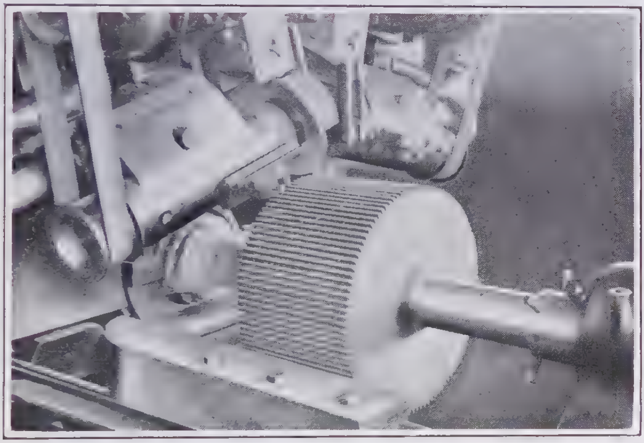


Fig. 188.—Large gear of wide face in position in the Maag gear grinder.

Another gear-tooth grinding machine is shown in Fig. 189. This machine was developed by the Pratt and Whitney Company, and described in the *American Machinist*, Oct. 11, 1923, from which the following is abstracted:

The design of this machine was based upon the following premises: first, that the high degree of accuracy of gear-tooth profiles necessary to the quiet performance of gears running at high speeds under heavy loads can be attained with the greatest certainty by the continued application of very light grinding cuts rather than by a lesser number of proportionately heavier cuts, because of the reduced tendency of the original errors to persist by reason of the inevitable "spring" of the grinding wheel, work, and work-carrying mechanism. Second, that multiple-tooth indexing would produce more accurate results under other similar

conditions than single-tooth indexing, because of the wider distribution and consequent reduction of errors due to wheel wear. Third, that in order to obtain a satisfactory rate of production under these conditions, the grinding cuts must be made in rapid succession. To whichever part it is applied—wheel or work—

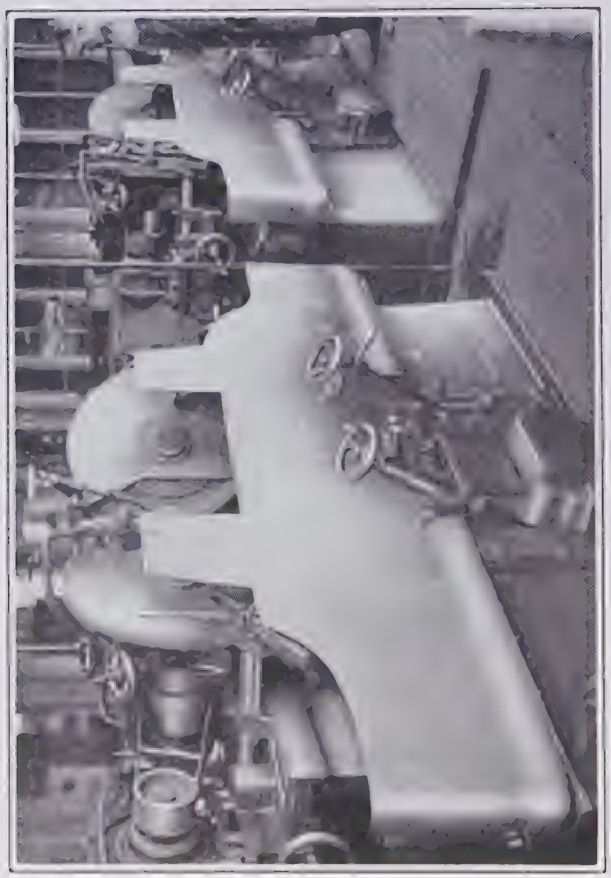


Fig. 189.—Pratt and Whitney gear grinder.

the movement must be continuous, without reversal of direction or pause for indexing.

Two grinding wheels of modified saucer shape are used. These wheels are about 24 in. in diameter. The active grinding surface of these wheels is a narrow band at the edge of each wheel and is a plane surface. Each wheel is independently mounted upon a head that may be adjusted in two directions and may also be

swiveled to any desired pressure angle. These movements are for adjustment only; the wheels in action have no movement other than that of rotation.

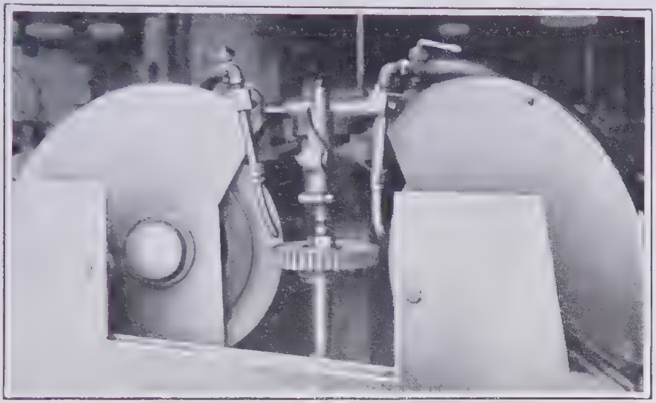


FIG. 190.—Work arbor mounted in swinging frame of Pratt and Whitney grinder.

The gear to be ground is held, with its axis vertical, between centers that are carried by a frame, which is so suspended from the column of the machine that, while the line of its centers is rigidly confined to parallel positions and it cannot move vertically, it is otherwise free to move in any direction. The mechanism that confines the movement of the gear to an arbitrary path is within the base of the machine below the swinging frame. A gear mounted on this swinging frame is shown in Fig. 190.

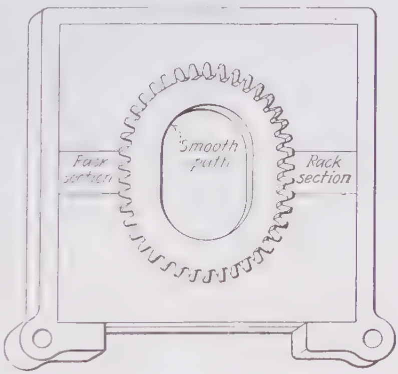


FIG. 191.—Work-arbor oscillation control unit of the Pratt and Whitney grinder.

The movement of this swinging frame and the rotation and indexing of the work spindle is controlled by a master gear mounted on the lower end of the work spindle in the swinging frame; this master gear meshes into a toothed path composed of straight rack sections inserted between the two halves of an internal gear. In addition, a smooth path of the same form operates with a smooth roll to prevent the master gear

from meshing too tightly with its toothed path. Figure 191 shows these two elements mounted in a cast-iron frame.

The grinding wheels are mounted over the rack or straight-line section of these controlling paths. As the work spindle is revolved, the master gear drives the spindle of the swinging frame around this path, the gear blank engaging first with one grinding wheel, as the master gear travels along the rack section on one side of the path, and then with the second wheel, as the master gear

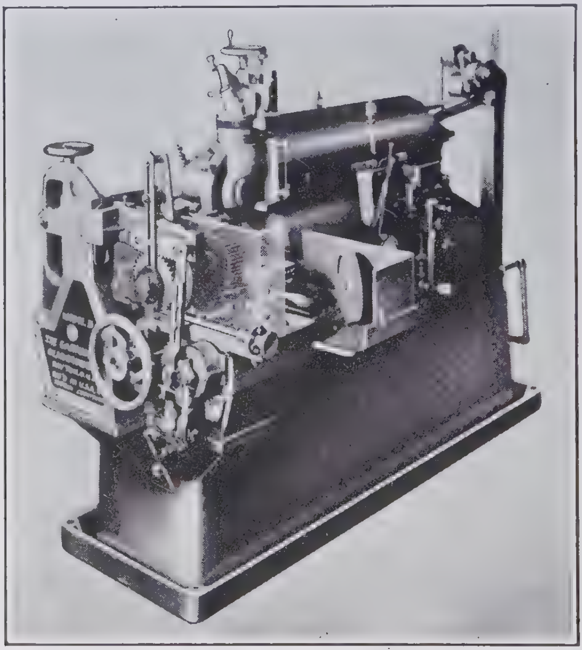


FIG. 192.—Garrison gear grinder employing a cone-shaped wheel carried on a ram.

travels along the rack section on the opposite side of the path. This motion is continuous, and the total number of teeth in the path is always such that there is no common divisor between this number and the number of teeth in the master gear. The master gear is of the same pitch and has the same number of teeth as the gear to be ground. In this way, multiple-tooth indexing is secured. For example, if the gear has 20 teeth and the path has 29, every ninth tooth in the gear will be ground successively.

The feed is obtained by moving the frame carrying the control paths in the direction that carries the gear-tooth profiles against the grinding faces of the wheels. This motion is also continuous until it reaches its final stop. The gear blank continues to travel through its orbit as many more times as necessary to work out the spring of the grinding wheels.

Another gear-tooth grinding machine that is quite different in many respects from any of the preceding ones is that developed by the Garrison Gear Grinder Company, illustrated in Fig. 192.

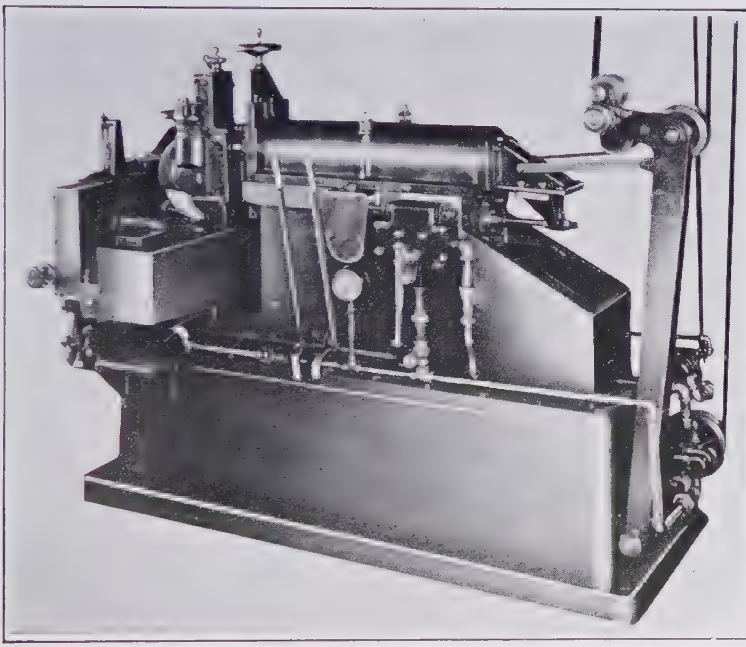


Fig. 193.—Hydraulically operated Garrison gear grinder.

This machine uses a cone-shaped wheel, which is mounted on a ram very similar to a ram of a shaping machine. This cone-shaped wheel travels through the tooth space, grinding both sides at once, while the gear blank is rolled back and forth under the wheel. In principle, this machine is very similar to the Sunderland and Maag gear-tooth shapers, which use rack-shaped cutters to generate the involute form.

In operation, the ram carrying the grinding wheel reciprocates through the tooth space while the work table traverses the blanks

past the grinding wheel. During each traverse of the table, the grinding wheel reciprocates in only one tooth space. The rolling motion of the gear blank is controlled by a master gear mounted at the end of the work spindle, which meshes with a master rack. At the end of each table movement, the master rack is lifted out of engagement with the master gear, and the work spindle is indexed one tooth space.

A further development of this machine employs a hydraulic oil system to control all motions of the machine except the rotation of the grinding wheel. This hydraulic machine is illustrated in Fig. 193.

Lapping of Gear Teeth.—The lapping of gear teeth at the present time is a tool-room or laboratory process. It is possible, however, that some form of this process may be developed in the future as a satisfactory production operation.

Hardened gears are sometimes run together under load with some form of abrasive introduced with the lubricant in an effort to smooth the surfaces and correct some of the errors. This process, however, does more grinding or crushing of the abrasive than it does to polish or lap the gear-tooth profiles. In order for an abrasive to work effectively, it must be rubbed along the surface to be lapped. The sliding action between meshing gear teeth is neither enough in amount nor uniform enough over the tooth profiles to obtain effective lapping action.

Another method is to mesh a hardened gear with a softer one of much wider face and to rotate them slowly together, with abrasive, under a suitable load, and at the same time to traverse the hardened gear rapidly across the face of the wide, soft gear, or lap. This process is very slow, sometimes requiring from 10 to 20 hours to finish a gear, but some remarkably accurate results have occasionally been thus secured.

Another method is to mount several gears on an arbor and to mesh them with a wide cast-iron rack of basic-rack form. These gears are then rolled slowly back and forth along the rack while traversing them rapidly across the face of the lap. In principle, this method is the same as the preceding one, with the substitution of a wide-faced rack for a wide-faced gear. In both of these two methods, the greatest problem is the maintenance of the laps, which is necessary in order to secure consistent results.

Another method, which is still in process of development, consists of the use of a lap in the form of an internal gear of the

same size and form as the gears to be lapped. The gear to be lapped is reciprocated through this internal gear and indexed at the end of each stroke, thus tending to distribute the errors uniformly about the circumference. There is a slight clearance, or backlash, between the gear and the lap, and the gear is held against one side of the lap teeth, thus requiring two operations to lap both sides of the gear teeth.

Burnishing of Gear Teeth.—In conclusion, mention should be made of the burnishing process for smoothing the tooth surfaces of unhardened gears. This process consists of running the cut gears under a heavy load with one or more hardened- and preferably ground-steel gears, thus cold-working the surfaces of the tooth profiles and producing a very smooth surface.

One of these processes, developed by the Pratt and Whitney Company, consists of the use of three hardened- and ground-steel gears or burnishers with the gear to be burnished set between them. The gear so confined squares itself on the teeth of the burnishers and requires no arbor, thus making possible a very rapid rate of production. A slight modification may be developed at the tips of the teeth of the burnished gear by making one or two of the burnishers with a slightly smaller normal pitch than that of the gear to be burnished. This process proves of particular advantage when used on gears cut with pinion-shaped cutters, as the sharp corners at the tips of the teeth of the hardened burnishers tend to remove any excessive fillets left in the cutting, in addition to the burnishing effect of the hardened tooth profiles of the burnishing gears.

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